Notes on this practice exam and the final exam
Dec. 2, 2011

• This practice exam is a collection of problems which are similar in style (but not necessarily in the choice of topics) to problems which may appear on the final exam. They are designed to be of a difficulty level roughly equal to, or slightly greater than, the final exam problems. The final exam will be shorter than the practice exam. I will publish solutions in a few days (around the middle of next week).

• The final exam will cover all the course material (the parts of the textbook we covered, the lecture notes, homework and its solutions, things discussed in class and discussion section). There will be an emphasis on material covered after the midterm. You do not need to memorize long proofs, but there will be some short proof questions that test your understanding of the concepts and basic reasoning ability.

• The final exam will take place on Friday, December 9 at 6 p.m. in the usual lecture room, Wellman 216 (see http://registrar.ucdavis.edu/csrg/chart.html).

• The final exam will be 2 hours long.

• The final exam will be a closed-book exam; no written material or electronic devices (calculators, cellphones etc.) will be allowed.

• Yvonne will hold a review session a day or two before the exam. (Details will be announced over email and on the class web page.)
Problem 1

You are given a system of 4 linear equations in 5 unknowns:

\[
\begin{align*}
2x_1 - 4x_2 + 2x_3 + 6x_5 &= 6 \\
x_3 - x_4 &= -1 \\
2x_1 - 4x_2 - x_3 + x_4 + 8x_5 &= 7 \\
x_3 + x_4 - 2x_5 &= 1
\end{align*}
\]

(a) Represent the system as an augmented matrix.

(b) Use the Gaussian elimination method to bring the augmented matrix to Reduced Row-Echelon Form (RREF).

(c) Use the RREF obtained in part (b) above to write the general form of the solution to the original system.

(d) Use the general form of the solution obtained in part (c) to write a specific solution to the system. That is, write specific numbers \(x_1, x_2, x_3, x_4, x_5\) that solve the system. Substitute the numbers into the system to verify that they actually satisfy the equations.
Problem 2

(a) Find a basis for $\mathbb{R}^2$ of eigenvectors of the matrix $\begin{pmatrix} 8 & -3 \\ -3 & 0 \end{pmatrix}$.

(b) Given real numbers $a, b, c$, find a formula for the eigenvalues $\lambda_1, \lambda_2$ of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Show that the eigenvalues are always real numbers, and characterize for what values of the parameters $a, b, c$ is it true that $\lambda_1 = \lambda_2$.

(c) Define what it means for a matrix to be diagonalizable.

(d) Prove that the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ (where $a, b, c$ are real numbers as above) is always diagonalizable.

**Hint:** divide into two cases according to whether $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$. 
Problem 3

(a) In the inner product space $\mathbb{R}^4$, let $U = \text{span}\{(1, 1, 1, 1), (1, 1, -1, -1)\}$. Recall that $U^\perp$ denotes the orthogonal complement of the space $U$. What equations must a vector $(x_1, x_2, x_3, x_4)$ satisfy to be in $U^\perp$?

(b) Find a basis for $U^\perp$.

(c) What is the dimension of the space $W = U + U^\perp$ (the sum of $U$ and $U^\perp$)? Explain how you know.

(d) Find an orthonormal basis in $\mathbb{R}^2$ containing the vector $(\frac{3}{5}, \frac{4}{5})$. 
Problem 4

Let $A$ be the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix}.$$ 

(a) Compute $A^{-1}$.

(b) Multiply the matrix you obtained in part (a) above by $A$ to check that it is indeed the inverse matrix of $A$.

(c) Find all the eigenvalues of $A$.

**Hint:** you should be able to answer this without any computations (or with a very short computation that will make the answer obvious).

(d) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$. 
Problem 5

(a) Compute the following determinants:

i. \( \det \begin{pmatrix} 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
2 & 1 & 2 & 1 \\
5 & 0 & 0 & 0 \end{pmatrix} \)

ii. \( \det \begin{pmatrix} 1 & 1 & 1 & 1 \\
0 & 2 & 2 & 2 \\
0 & 0 & 3 & 3 \\
0 & 0 & 0 & 1 \end{pmatrix} \)

iii. \( \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \end{pmatrix} \) (note: the answer may depend on \( n \))

(b) Let \( A, B \) be square matrices of order \( n \). Assume that \( B \) is invertible. Prove that

\[ \det(BAB^{-1}) = \det(A). \]

(c) A square matrix \( A \) of order \( n \) is called anti-symmetric if it satisfies the condition

\[ A^T = -A \]

For example, the matrix

\[
\begin{pmatrix}
0 & 5 & -1 \\
-5 & 0 & 2 \\
1 & -2 & 0
\end{pmatrix}
\]

is anti-symmetric. Prove that if \( n \) is an odd number and \( A \) is an anti-symmetric matrix of order \( n \) then \( A \) is not invertible. (Hint: use determinants.)
Problem 6

(a) If a linear operator $T : V \to V$ has two eigenvectors $v_1, v_2$. Assume that the associated eigenvalues $\lambda_1$ and $\lambda_2$ are distinct. Are $v_1, v_2$ necessarily linearly independent? Prove that they are, or give an example that shows they don’t have to be.

(b) Let $V$ be a vector space with $\dim(V) = 3$, and let $T : V \to V$ be a linear operator on $V$. Assume that $v_1, v_2, v_3$ are eigenvectors of $T$, with associated eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 1.$$ 

Assume also that $v_1$ and $v_2$ are linearly independent. Prove that \{v_1, v_2, v_3\} is a basis of $V$.

**Hint:** If $v_1, v_2, v_3$ are linearly dependent, show that you can find linearly dependent eigenvectors $u, w$ such that $T(u) = \lambda_1 u$ and $T(w) = \lambda_3 w$, and explain why this leads to a contradiction.