Solutions to Midterm Exam 1

1 Write the sum $1+2+3+4+\ldots+29+30$ in $\sum$ notation, and evaluate it.

Solution: $1+2+\ldots+30 = \sum_{k=1}^{30} k = \frac{30\times 31}{2} = 465$.

2 Find the total area of the regions bounded between the graph of $y=\sin(x)$ and the x-axis (both below and above the axis) in the interval $[0,2\pi]$.

Solution: Recall that $\sin x \geq 0$ if x is in $[0,\pi]$, and $\sin x \leq 0$ if x is in $[\pi,2\pi]$. So the total area is

$$\begin{aligned} \int_0^{2\pi} |\sin x|\,dx &= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)dx = -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi} \\ &= (1-(-1)) + (1-(-1)) = 4. \end{aligned}$$

A common mistake was to integrate $\sin x$ instead of $|\sin x|$, which gives 0. This clearly does not make sense, since the area has to be positive. The moral is that it is always good to do a “sanity check” on your answers.

3 If $f(x)$ is integrable and $\int_0^4 f(x)\,dx = 10$, $\int_2^4 f(x)\,dx = 6$, what is $\int_0^1 f(2x)\,dx$?

Solution: Using the substitution $u=2x$, we see that

$$\int_0^1 f(2x)\,dx = \int_0^2 f(u)\,\frac{du}{2} = \frac{1}{2}\int_0^2 f(u)\,du = \frac{1}{2}\int_0^2 f(x)\,dx.$$

But we also know that $10 = \int_0^4 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx = \int_0^2 f(x)\,dx + 6$, so $\int_0^2 f(x)\,dx = 10-6 = 4$ and, consequently, $\int_0^1 f(2x)\,dx = 4/2 = 2$.

4 Evaluate the following definite integrals:

(a) $\int_{-1}^{3} \sqrt{1+x}\,dx$

Solution: Make the substitution $u=1+x$. By the substitution rule, this gives

$$\int_{-1}^{3} \sqrt{1+x}\,dx = \int_0^4 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^4 = \frac{2}{3}4\sqrt{4} = \frac{16}{3}.$$

(b) $\int_0^2 xe^{x^2}\,dx$

Solution: Substitute $u = x^2$, to get that $du = 2x\,dx$, so

$$\int_0^2 xe^{x^2}\,dx = \frac{1}{2}\int_0^{2^2} e^u\,du = \frac{1}{2}e^u\Big|_0^4 = \frac{e^4 - e^0}{2} = \frac{e^4-1}{2}.$$

5 (a) Compute the indefinite integral $\int 10\cos(x)\sin^4(x)\,dx$.

Solution: Make the substitution $u = \sin x$. The answer (which is easily verified by differentiation) is $2\sin^5 x + C$.

(Note: A common mistake here was to omit the arbitrary constant C.)

(b) Find an anti-derivative for $f(x) = \frac{1}{(x-2)^4}$.

Solution: $F(x) = -\frac{1}{3(x-2)^3}$ is an anti-derivative (guess it, or make the substitution $u = x - 2$ and integrate).

Note that here the constant C is not necessary (although adding it still leads to a correct answer), since you were only asked to produce an example (one out of the infinitely many possibilities, one for each choice of C) of a function whose derivative is $f(x)$.