Solutions to Midterm Exam 2

1 (25 points) The region bounded between the x-axis and the curve $y = 1 - x^2$ is revolved around the line $y = -1$. Compute the volume of the resulting solid. **(Note that the solid has a "hole".)**

Solution. This is a solid of revolution of "washer" type, with outer radius

$$R(x) = (1 - x^2) - (-1) = 2 - x^2,$$

and inner radius $r(x) = 0 - (-1) = 1$. The range for the x-coordinate is found by intersecting the curve $y = 1 - x^2$ with the x-axis $y = 0$, which gives $x = -1, +1$. Therefore the volume is computed as

$$\begin{aligned} V &= \int_{-1}^{1} \pi \left[(2 - x^2)^2 - 1^2 \right] \, dx = \pi \int_{-1}^{1} \left(3 - 4x^2 + x^4 \right) \, dx \\ &= \pi \left(3 \times 2 - \tfrac{4}{3} x^3 \Big|_{-1}^{1} + \tfrac{1}{5} x^5 \Big|_{-1}^{1} \right) = \pi \left(6 - \tfrac{8}{3} + \tfrac{2}{5} \right) = \frac{56\,\pi}{15}. \end{aligned}$$

2 (25 points) A bucket weighing 10 pounds is pulled up to the roof of a 30-foot high building using a rope and pulley. The rope itself has a weight density of 0.3 lbs/ft. Compute the amount of work that is required, in lb-ft units.

Solution. The force $F(y)$ that the person pulling the rope has to apply as a function of the height y that the bucket was already lifted is given by

$$F(y) = 10 + 0.3(30 - y),$$

where 10 is the constant weight of the bucket, and $0.3(30 - y)$ is the weight of the part of the rope that was still not hoisted up on the roof. Therefore the work is equal to

$$\begin{aligned} W &= \int_0^{30} F(y) \, dy = \int_0^{30} 10 + 0.3(30 - y) \, dy = 300 + 0.3 \frac{-(30 - y)^2}{2} \Big|_0^{30} \\ &= 300 + \frac{0.3 \times 30^2}{2} = 300 + 135 = 435. \end{aligned}$$

3 (25 points) A thin rod lies on the x-axis between $x = 0$ and $x = 1$. Its density is given by $f(x) = \frac{1}{x+1}$.

(a) Compute the mass of the rod.

(b) Compute the moment of the rod about 0.

Solution. The mass and the moment about 0 are given, respectively, by

$$\begin{aligned} M &= \int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^1 = \ln(2) - \ln(1) = \ln(2), \\ M_0 &= \int_0^1 \frac{1}{x+1} x\,dx = \int_0^1 \left(1 - \frac{1}{x+1}\right) dx = 1 - \int_0^1 \frac{1}{x+1}\,dx = 1 - \ln(2). \end{aligned}$$

4 (25 points) Find the arc length of the curve given in parametric form by

$$x = t - e^{2t}, \qquad y = 2\sqrt{2}e^t, \quad 1 \le t \le 2.$$

Solution. The arc length formula for a curve in parametric form is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

In our case the integrand is

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{(1-2e^{2t})^2 + \left(2\sqrt{2}e^t\right)^2} = \sqrt{1 + 4e^{2t} + 4e^{4t}} \\ &= \sqrt{(1+2e^{2t})^2} = \left|1 + 2e^{2t}\right| = 1 + 2e^{2t}. \end{aligned}$$

So the arc length is easily computed to be

$$L = \int_1^2 \left(1 + 2e^{2t}\right) dt = 1 + e^4 - e^2.$$