Solutions to practice questions for midterm exam 2

1. Let C denote the curve $y = 2 - |x|, \ \ -1 \le x \le 1$.

(a) Compute the volume of the solid of revolution formed by revolving the region bounded between C and the x-axis about the x-axis.

(b) Compute the surface area of the surface of revolution of C about the x-axis.

(c) Compute the arc length of C.

Solution. In all three cases, we can use symmetry by computing the answer for the part of the curve in the range $0 \le x \le 1$ and multiplying it by 2. This gives

$$\begin{aligned}
\text{(a)} \quad V &= \int_{-1}^{1} \pi y^2\,dx = 2\int_0^1 \pi y^2\,dx = 2\pi \int_0^1 (2-x)^2\,dx = 2\pi \frac{-(2-x)^3}{3}\Big|_0^1 \\
&= \frac{2\pi}{3}(2^3 - 1^3) = \frac{14\pi}{3}, \\
\text{(c)} \quad L &= 2\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\int_0^1 \sqrt{2}\,dx = 2\sqrt{2}, \\
\text{(b)} \quad A &= 2\int_0^1 2\pi(2-x)\sqrt{2}\,dx = \ldots = 6\sqrt{2}\pi.
\end{aligned}$$

2. The parametric curve

$$x = \frac{2}{3}t^{3/2}, \quad y = 2\sqrt{t}, \qquad 0 \le t \le \sqrt{3}$$

is revolved about the y-axis. Find the area of the resulting surface.

Solution. The differential arc-length is given by

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{t + t^{-1}}\, dt = \sqrt{\frac{t^2+1}{t}}\, dt,$$

so the surface area is

$$A = \int_0^{\sqrt{3}} 2\pi x(t) \sqrt{\frac{t^2+1}{t}}\, dt = \int_0^{\sqrt{3}} 2\pi \cdot \frac{2}{3} \cdot t\sqrt{t^2+1}\, dt$$

This integral can be computed by making the substitution $u = t^2 + 1$, leading to

$$A = \frac{2\pi}{3} \int_1^4 \sqrt{u}\, du = \ldots = \frac{28\pi}{9}.$$

3. A thin rod of length 2 meters is made of a material whose composition changes along the length of the rod. The density at distance x from the left end of the rod is $f(x) = 100 + x^2/200$ grams per centimeter. Find the total mass of the rod and its center of mass.

Solution.

$$\begin{aligned}
\text{Mass:} \quad M &= \int_0^{200} f(x)\, dx = \int_0^{200} \left(100 + \frac{x^2}{200}\right) dx = \ldots = \frac{100000}{3} \text{ gr} \\
&= 33\tfrac{1}{3} \text{ kg} \\
\text{Moment:} \quad M_0 &= \int_0^{200} x\, f(x)\, dx = \int_0^{200} \left(100x + \frac{x^3}{200}\right) dx = \ldots \\
&= 4 \times 10^6 \text{ gr} \times \text{cm}. \\
\text{Center of mass:} \quad \overline{x} &= \frac{M_0}{M} = 120 \text{ cm} = 1.2 \text{ m}.
\end{aligned}$$

4. (a) A spring satisfies Hooke's law with an elastic constant $k_1 = 20$ lb/ft. Compute the work needed to stretch the spring from its original length of 3 feet to 4 feet. Specify the units for the answer.

(b) Compute the answer to the same question, under the additional assumption that after stretching the spring by 0.5 feet, the spring suffers a partial breakdown of its elastic material, causing the elastic constant to drop in value to $k_2 = 8$ lb/ft. In other words, the spring's resistive force $F(x)$ as a function of the amount x by which it was stretched is given by

$$F(x) = \begin{cases} k_1 x & \text{if } x < 0.5, \\ k_2 x & \text{if } x \geq 0.5. \end{cases}$$

Solution.

$$\begin{aligned} \text{(a)} \quad W_1 &= \int_0^1 20x\,dx = 10 \text{ lb-ft}, \\ \text{(b)} \quad W_2 &= \int_0^1 F(x)\,dx = \int_0^{0.5} 20x\,dx + \int_{0.5}^1 8x\,dx = 2.5 + 3 \text{ lb-ft} = 5.5 \text{ lb-ft}. \end{aligned}$$

5. A water tank is shaped like the surface of revolution about the y-axis of the curve $y = 3x^2$, $0 \leq x \leq 2$, (both x and y have meter units). It is filled with water, whose density is 1000 kg/m^3, up to a height of 8 meters.

(a) Compute the total mass of the water in the tank. (Hint: there is a simple relation between the mass and the volume...)

(b) Compute the amount of work that needs to be expended to pump all the water out over the sides of the tank. Note that you need to know the height of the tank, and use the fact that the Earth's gravity pulls down a mass of 1 kilogram with a force equal (roughly) to 9.8 Newtons – this value is the Earth's gravitational constant, commonly denoted g.

Solution. The mass is the volume of the region in space containing the water (which is a solid of revolution) times the density of water:

$$M = 1000 \int_0^8 \pi x^2\,dy = 1000\pi \int_0^8 (\sqrt{y/3})^2\,dy = \frac{1000\pi}{3} \int_0^8 y\,dy = \frac{32000\pi}{3} \text{ kg.}$$

To compute the work, the factor $\pi x^2\,dy$ in the integral above needs to be multiplied by the Earth's gravitational constant $g = 9.8$ N/kg (to get the force with which gravity pulls down that layer of water) and then further multiplied by the distance in the vertical direction that that layer has to be transported to pump it over the side of the tank, which is equal to $12 - y$ (since the height of the tank is $y = 3 \times 2^2 = 12$). This gives:

$$W = \frac{9800\pi}{3} \int_0^8 y(12-y)\,dy = \ldots = \frac{9800\pi}{3} \cdot \frac{640}{3} = \frac{6,272,000\pi}{9} = 2.18934 \times 10^6 \text{ J.}$$

(∗) **6.** The famous Gateway Arch in Saint Louis, Missouri is described mathematically (ignoring its third dimension) as the two-dimensional region located above the x-axis and bounded between the two curves

$$\begin{aligned} y_{\text{outer}} &= H - \frac{a}{2}\left(e^{x/a} + e^{-(x/a)} - 2\right), \\ y_{\text{inner}} &= h - \frac{c}{2}\left(e^{x/c} + e^{-(x/c)} - 2\right). \end{aligned}$$

See Figure 1 below. When x and y are measured in feet, the parameters H, h, a, c are given by

$$\begin{aligned} H &= 630 \text{ ft}, \quad \text{(the height of the arch)}, \\ h &= 615 \text{ ft}, \\ a &= 127.7 \text{ ft}, \\ c &= 101.4 \text{ ft}. \end{aligned}$$

It can be checked that y_{outer} is positive exactly when $-L < x < L$, where $L \approx 315$ ft (in other words the span of the arch is $2L \approx 630$ ft), and similarly y_{inner} is positive when $-K < x < K$, where $K \approx 263$ ft (i.e., the "inner span" is 526 feet).

(a) Find the arc length of the outer curve.

(b) Find the area of the arc (again treating it as a two-dimensional object). Note that in the ranges where $K \le |x| \le L$ the inner curve lies below the x-axis, so those ranges need to be treated separately from the range $-K \le x \le K$.

Solution. (a) To compute the arc length, we compute the arc length element $ds = \sqrt{1 + (dy_{\text{outer}}/dx)^2}\, dt$

$$
\begin{aligned}
y_{\text{outer}} &= H - \frac{a}{2}\left(e^{x/a} + e^{-x/a} - 2\right), \\
\frac{dy_{\text{outer}}}{dx} &= -\tfrac{1}{2}\left(e^{x/a} - e^{-x/a}\right), \\
\left(\frac{dy_{\text{outer}}}{dx}\right)^2 &= \tfrac{1}{4}\left(e^{2x/a} + e^{-2x/a} - 2\right), \\
\sqrt{1 + \left(\frac{dy_{\text{outer}}}{dx}\right)^2} &= \sqrt{\tfrac{1}{4}\left(e^{2x/a} + e^{-2x/a} + 2\right)} = \tfrac{1}{2}\left(e^{x/a} + e^{-x/a}\right).
\end{aligned}
$$

Therefore the arc length is

$$
\text{length} = \int_{-L}^{L} \tfrac{1}{2}\left(e^{x/a} + e^{-x/a}\right)\, dt = \frac{a}{2} e^{x/a}\Big|_{-L}^{L} - \frac{a}{2} e^{-x/a}\Big|_{-L}^{L} = a\left(e^{L/a} - e^{-L/a}\right) \approx 1494 \text{ ft.}
$$

(b) Using the suggestion in the question, the area between the inner and outer curves and above the x-axis is computed as

$$
\begin{aligned}
A &= \int_{-L}^{-K} y_{\text{outer}}\, dx + \int_{-K}^{K} \left(y_{\text{outer}} - y_{\text{inner}}\right)\, dx + \int_{K}^{L} y_{\text{outer}}\, dx \\
&= 2\int_{0}^{K} \left(y_{\text{outer}} - y_{\text{inner}}\right)\, dx + 2\int_{K}^{L} y_{\text{outer}}\, dx
\end{aligned}
$$

(the last equality is a consequence of the symmetry of the arch around the y-axis). This is easily computed with a calculator to give

$$
A = 2 \times 16356.5 + 2 \times 6914.56 \text{ ft}^2 = 46542.1 \text{ ft}^2.
$$