Math 21B UC Davis, Winter 2010 Dan Romik

Solutions to Practice Exam 1

1 Evaluate $\sum\limits_{k=1}^{20}(2k-2)$.

Solution. Using the properties of Σ-notation sums and the formula for the sum of the first n integers, we get:

$$\begin{aligned}\sum_{k=1}^{20}(2k-2) &= \sum_{k=1}^{20}2k-\sum_{k=1}^{20}2 = 2\sum_{k=1}^{20}k-\sum_{k=1}^{20}2 \\ &= 2\times\frac{20(20+1)}{2}-2\times 20 = 420-40=380.\end{aligned}$$

2 If $f(x)$ is continuous and $\int\limits_0^{25} f(x)\,dx = 8$, what is $\int\limits_0^5 f(x^2)x\,dx$?

Solution. We perform the substitution $u = x^2$. Using the substitution rule for definite integrals, this gives that

$$\int_0^5 f(x^2)x\,dx = \frac{1}{2}\int_0^5 f(x^2)(2x)\,dx = \frac{1}{2}\int_0^{25} f(u)\,du = \frac{1}{2}\int_0^{25} f(x)\,dx = \frac{1}{2}\times 8 = 4.$$

3 Find the area of the region bounded between the curves $y = 3x$ and $y = x^2$.

Solution. First, it is a good idea to sketch a diagram showing the graphs of the two curves and the region bounded between them. Figure 1 shows the result. Now we find the x-ordinates of the points of intersection of the two curves. This leads to the equation $x^2 = 3x$, which has the two solutions $x = 0$ and $x = 3$. Finally, the area is computed by integrating the difference of the two curves between the two points

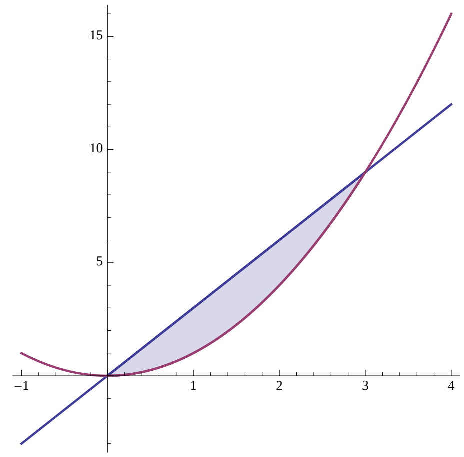

Figure 1: The curves $y = 3x$ and $y = x^2$ and the region bounded between them.

of intersection (taking care to subtract the smaller one from the larger one and not vice versa, to get the answer with a positive sign):

$$\begin{aligned} \text{Area} &= \int_0^3 (3x - x^2)\,dx = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=3} \\ &= \left(\frac{3 \times 3^2}{2} - \frac{3^3}{3}\right) - \left(\frac{3 \times 0^2}{2} - \frac{0^3}{3}\right) = 13.5 - 9 = 4.5. \end{aligned}$$

4 (30 points) Evaluate the following definite integrals:

(a) $\int\limits_0^3 x^2(1 + 4x)\,dx$

(b) $\int\limits_{-2}^1 (2x - |x|)\,dx$

(c) $\int\limits_0^1 \frac{x}{1+x^4}\,dx$

Solution.

$$\begin{aligned}\int_0^3 x^2(1+4x)\,dx &= \int_0^3 (x^2+4x^3)\,dx = x^3/3\Big|_0^3 + x^4\Big|_0^3 \\ &= (27/3-0/3)+(3\times3\times3\times3-0) = 9+81 = 90.\end{aligned}$$

For the second integral, since $|x|$ is either $-x$ or x according as whether x is negative or positive, we compute the integral as a sum of two integrals over the intervals $[-2,0]$ and $[0,1]$:

$$\begin{aligned}\int_{-2}^{1} (2x-|x|)\,dx &= \int_{-2}^0 (2x-(-x))\;dx + \int_0^1 (2x-x)\,dx \\ &= \int_{-2}^0 3x\,dx + \int_0^1 x\,dx = 3x^2/2\Big|_{-2}^0 + x^2/2\Big|_0^1 \\ &= (0-3\times4/2)+(0.5-0) = -5.5.\end{aligned}$$

For the third integral, we make the substitution $u=x^2$:

$$\begin{aligned}\int_0^1 \frac{x}{1+x^4}\,dx &= \frac{1}{2}\int_0^1 \frac{1}{1+(x^2)^2}\,(2x)dx = \frac{1}{2}\int_0^{1^2}\frac{1}{1+u^2}\,du \\ &= \frac{1}{2}\Big(\tan^{-1}(1)-\tan^{-1}(0)\Big) = \frac{1}{2}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{8}.\end{aligned}$$

5 (20 points) Compute the following indefinite integrals:
(Example: $\int 10x^4\,dx = 2x^5 + C$. Don't forget the "$C$"...)

(a) $\int x^2\sin(x^3)\,dx$

(b) $\int (3x+1)^{99}\,dx$

Solution. **(a)** Make the substitution $u=x^3$, to get $du = 3x^2\,dx$, so

$$\int x^2\sin(x^3)\,dx = \frac{1}{3}\int \sin(u)\,du = \frac{1}{3}(-\cos u)+C = -\frac{1}{3}\cos(x^3)+C.$$

(b) Substitute $t=3x+1$ to get

$$\int (3x+1)^{99}\,dx = \frac{1}{3}\int t^{99}\,dt = \frac{1}{300}t^{100}+C = \frac{1}{300}(3x+1)^{100}+C.$$