Solutions to Practice Exam 1

1 Evaluate $\sum_{k=1}^{20} (2k - 2)$.

**Solution.** Using the properties of $\Sigma$-notation sums and the formula for the sum of the first $n$ integers, we get:

$$
\sum_{k=1}^{20} (2k - 2) = \sum_{k=1}^{20} 2k - \sum_{k=1}^{20} 2 = 2 \sum_{k=1}^{20} k - \sum_{k=1}^{20} 2
= 2 \times \frac{20(20 + 1)}{2} - 2 \times 20 = 420 - 40 = 380.
$$

2 If $f(x)$ is continuous and $\int_0^{25} f(x) \, dx = 8$, what is $\int_0^5 f(x^2) x \, dx$?

**Solution.** We perform the substitution $u = x^2$. Using the substitution rule for definite integrals, this gives that

$$
\int_0^5 f(x^2) x \, dx = \frac{1}{2} \int_0^5 f(u) \, du = \frac{1}{2} \int_0^{25} f(u) \, du = \frac{1}{2} \int_0^{25} f(x) \, dx = \frac{1}{2} \times 8 = 4.
$$

3 Find the area of the region bounded between the curves $y = 3x$ and $y = x^2$.

**Solution.** First, it is a good idea to sketch a diagram showing the graphs of the two curves and the region bounded between them. Figure 1 shows the result. Now we find the $x$-ordinates of the points of intersection of the two curves. This leads to the equation $x^2 = 3x$, which has the two solutions $x = 0$ and $x = 3$. Finally, the area is computed by integrating the difference of the two curves between the two points...
of intersection (taking care to subtract the smaller one from the larger one and not vice versa, to get the answer with a positive sign):

\[
\text{Area} = \int_0^3 (3x - x^2) \, dx = \left( \frac{3x^2}{2} - \frac{x^3}{3} \right) \bigg|_{x=0}^{x=3} \\
= \left( \frac{3 \times 3^2}{2} - \frac{3^3}{3} \right) - \left( \frac{3 \times 0^2}{2} - \frac{0^3}{3} \right) = 13.5 - 9 = 4.5.
\]

4 (30 points) Evaluate the following definite integrals:

(a) \[ \int_0^3 x^2(1 + 4x) \, dx \]

(b) \[ \int_{-2}^1 (2x - |x|) \, dx \]

(c) \[ \int_0^1 \frac{x}{1 + x^2} \, dx \]
Solution.

\[ \int_0^3 x^2(1 + 4x) \, dx = \int_0^3 (x^2 + 4x^3) \, dx = x^3/3 \bigg|_0^3 + x^4 \bigg|_0^3 = (27/3 - 0/3) + (3 \times 3 \times 3 \times 3 - 0) = 9 + 81 = 90. \]

For the second integral, since \(|x|\) is either \(-x\) or \(x\) according as whether \(x\) is negative or positive, we compute the integral as a sum of two integrals over the intervals \([-2, 0]\) and \([0, 1]\):

\[ \int_{-2}^{1} (2x - |x|) \, dx = \int_{-2}^{0} (2x - (-x)) \, dx + \int_{0}^{1} (2x - x) \, dx \]
\[ = \int_{-2}^{0} 3x \, dx + \int_{0}^{1} x \, dx = 3x^2/2 \bigg|_{-2}^{0} + x^2/2 \bigg|_{0}^{1} \]
\[ = (0 - 3 \times 4/2) + (0.5 - 0) = -5.5. \]

For the third integral, we make the substitution \(u = x^2\):

\[ \int_{0}^{1} \frac{x}{1 + x^4} \, dx = \frac{1}{2} \int_{0}^{1} \frac{1}{1 + (x^2)^2} (2x) \, dx = \frac{1}{2} \int_{0}^{1} \frac{1}{1 + u^2} \, du \]
\[ = \frac{1}{2} \left( \tan^{-1}(1) - \tan^{-1}(0) \right) = \frac{1}{2} \left( \frac{\pi}{4} - 0 \right) = \frac{\pi}{8}. \]

5 (20 points) Compute the following indefinite integrals:
(Example: \( \int 10x^4 \, dx = 2x^5 + C \). Don’t forget the “C”...)

(a) \( \int x^2 \sin(x^3) \, dx \)

(b) \( \int (3x + 1)^{99} \, dx \)

Solution. (a) Make the substitution \(u = x^3\), to get \(du = 3x^2 \, dx\), so

\[ \int x^2 \sin(x^3) \, dx = \frac{1}{3} \int \sin(u) \, du = \frac{1}{3} (-\cos u) + C = -\frac{1}{3} \cos(x^3) + C. \]

(b) Substitute \(t = 3x + 1\) to get

\[ \int (3x + 1)^{99} \, dx = \frac{1}{3} \int t^{99} \, dt = \frac{1}{300} t^{100} + C = \frac{1}{300} (3x + 1)^{100} + C. \]