Solutions to Sample Exam

Question 1 Compute the following integrals:

(a) $$\int_0^\pi (x^2 + \sin(x))\,dx$$

$$\int_0^\pi (x^2 + \sin(x))\,dx = \left(\frac{1}{3}x^3 - \cos(x)\right)\Big|_0^\pi = \frac{\pi^3}{3} - (-1) - (0-1) = \frac{\pi^3}{3} + 2$$

(b) $$\int \sec^2(x)\,dx$$

$$\int \sec^2(x)\,dx = \tan(x) + C \qquad \text{(standard integral)}$$

(c) $$\int \cos(2x-1)\,dx$$

$$\int \cos(2x-1)\,dx = \frac{1}{2}\sin(2x-1) + C \qquad \text{(use substitution } u = 2x-1\text{)}$$

(d) $$\int \frac{(1+\sqrt{x})^7}{\sqrt{x}}\,dx$$

$$\begin{aligned}\int \frac{(1+\sqrt{x})^7}{\sqrt{x}}\,dx &= 2\int \frac{(1+\sqrt{x})^7}{2\sqrt{x}}\,dx = 2\int (1+u)^7\,du = \frac{1}{4}(1+u)^8 + C \\ &= \frac{1}{4}\left(1+\sqrt{x}\right)^8 + C \quad \text{(use the substitution } u = \sqrt{x}\text{)}\end{aligned}$$

(e) $$\int_1^b (e^y + e^{-y})\,dy \quad \text{(note: the answer is a function of } b\text{)}$$

$$\int_1^b (e^y + e^{-y})\,dy = \left(e^y - e^{-y}\right)\Big|_1^b = e^b - e^{-b} - e + e^{-1}$$

(f) $$\int_0^2 x e^{x^2}\,dx$$

$$\int_0^2 x e^{x^2}\,dx = \int_0^4 \frac{1}{2}e^u\,du = \frac{1}{2}\left(e^4 - e^0\right) = \frac{1}{2}\left(e^4 - 1\right) \quad \text{(use the substitution } u = x^2\text{)}$$

Question 2 Compute the arc length of the curve $y = x^3/6 + 1/(2x)$ from $x = 2$ to $x = 3$.

Solution. The arc length is given by $\int_2^3 \sqrt{1 + (y')^2}\, dx$. First we simplify the integrand:

$$\begin{aligned} y' &= \frac{x^2}{2} - \frac{1}{2x^2}, \\ 1 + (y')^2 &= 1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 = 1 + \left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right) \\ &= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, \\ \sqrt{1 + (y')^2} &= \left|\frac{x^2}{2} + \frac{1}{2x^2}\right| = \frac{x^2}{2} + \frac{1}{2x^2} \quad \text{(absolute value is redundant for } 2 \le x \le 3\text{)}. \end{aligned}$$

Now proceed to compute the arc length:

$$\int_2^3 \sqrt{1 + (y')^2}\, dx = \int_2^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left(\frac{x^3}{6} - \frac{1}{2x}\right)\Big|_2^3 = \left(\frac{9}{2} - \frac{1}{6}\right) - \left(\frac{4}{3} - \frac{1}{4}\right) = \frac{13}{4}.$$

Question 3 Compute the following integrals using integration by parts or any other technique.

(a) $\int \ln x\, dx$: Use integration by parts with $u = \ln x$, $v = x$ (so that $dv = 1 \cdot dx$):

$$\begin{aligned} \int \ln x\, dx &= \int \ln x \cdot 1\, dx = x \cdot \ln x - \int x \frac{d}{dx}(\ln x)\, dx = x \ln x - \int 1\, dx \\ &= x \ln x - x + C \end{aligned}$$

(b) $\int x \ln x\, dx$: Use integration by parts with $u = \ln x$, $v = \frac{1}{2}x^2$:

$$\begin{aligned} \int x \ln x\, dx &= \tfrac{1}{2}x^2 \cdot \ln x - \int \tfrac{1}{2}x^2 \frac{d}{dx}(\ln x)\, dx = \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x\, dx \\ &= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C \end{aligned}$$

(c) $\int_0^\pi x^2 \sin(x)\, dx$: Use integration by parts twice:

$$\begin{aligned} \int_0^\pi x^2 \sin(x)\, dx &= x^2(-\cos x)\Big|_0^\pi - \int_0^\pi 2x(-\cos x)\, dx \\ &= (\pi^2 \cdot 1 - 0) + 2\int_0^\pi x \cos x\, dx \\ &= \pi^2 + 2\left(x \sin x\Big|_0^\pi - \int_0^\pi \sin x\, dx\right) \\ &= \pi^2 + 2(0 - 0) + 2\cos x\Big|_0^\pi = \pi^2 - 4 \end{aligned}$$

(d) $$\int \frac{x^2-2}{x(x-1)(x-2)}\,dx:$$

Solution: We look for a partial fraction decomposition for the integrand, of the form
$$\frac{x^2-2}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$
where A, B, C are unknown coefficients. They can be found in two ways. The easiest is to use the "cover-up" method (which is applicable here since the denominator of the rational function is a product of linear factor, each appearing with power 1):

$$\begin{aligned} A &= \left.\frac{x^2-2}{(x-1)(x-2)}\right|_{x=0} = \frac{-2}{(-1)(-2)} = -1, \quad \text{(the "}x\text{" factor is covered up)} \\ B &= \left.\frac{x^2-2}{x(x-2)}\right|_{x=1} = \frac{1-2}{1\cdot(-1)} = 1, \quad \text{(the "}x-1\text{" factor is covered up)} \\ C &= \left.\frac{x^2-2}{x(x-1)}\right|_{x=2} = \frac{2^2-2}{2\cdot 1} = 1, \quad \text{(the "}x-2\text{" factor is covered up).} \end{aligned}$$

The second method involves multiplying out the partial fraction decomposition above by $x(x-1)(x-2)$, to get the equation

$$x^2 - 2 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1)$$

and then equate coefficients of powers of x on both sides to get the equations

$$\begin{aligned} A + B + C &= 1, \\ 3A + 2B + C &= 0, \\ 2A &= -2. \end{aligned}$$

These equations can easily be solved to give the same answers $A = -1, B = C = 1$. Having found the coefficients, we can compute the integral:

$$\begin{aligned} \int \frac{x^2-2}{x(x-1)(x-2)} &= \int \left(-\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2}\right) dx \\ &= -\ln|x| + \ln|x-1| + \ln|x-2| + C. \end{aligned}$$

Question 4 Estimate the value of the definite integral $\int_0^2 3x^2\,dx$ in two ways: (a) Using the trapezoidal rule, and (b) using Simpson's rule. In both cases use a partition of the interval $[0, 2]$ into $n = 4$ sub-intervals of equal length. Compare the results to the correct value of the integral $\int_0^2 3x^2\,dx$ and determine which one is closer to this value.

Solution. The partition points are $x_0 = 0, x_1 = 1/2, x_2 = 1, x_3 = 3/2, x_4 = 2$. $\Delta x = 1/2$, so we have the estimates:

$$\begin{aligned} I_T &= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right) \\ &= \frac{1}{4}\left(3\cdot 0^2 + 2\cdot 3(1/2)^2 + 2\cdot 3(1^2) + 2\cdot 3(3/2)^2 + 3(2^2)\right) \\ &= \frac{3}{4}\left(0 + \frac{1}{2} + 2 + \frac{9}{2} + 4\right) = \frac{33}{4}, \\ I_S &= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right) \\ &= \frac{1}{6}\left(3\cdot 0^2 + 4\cdot 3(1/2)^2 + 2\cdot 3(1^2) + 4\cdot 3(3/2)^2 + 3(2^2)\right) \\ &= \frac{1}{2}(0 + 1 + 2 + 9 + 4) = 8. \end{aligned}$$

To determine which of the estimates is closer to the true value, we compute it:

$$\int_0^2 3x^2\,dx = x^3\Big|_0^2 = 2^3 - 0 = 8.$$

Simpson's rule actually gives the precise answer in this case! (Side remark: this will always be true when applying Simpson's rule to numerically integrate quadratic polynomials). Therefore, trivially, the Simpson's rule is more precise in this case than the trapezoidal rule.

Question 5 Compute the volume of the solid formed by revolving around the x-axis the region bounded between the x-axis and the curve $y = x - x^2/4$.

Solution. First, we find the range of integration by finding where the curve intersects the x-axis. This gives the equation $y = x - x^2/4 = 0$, which has the solutions $x = 0, 4$. Therefore the integration is on the interval $[0, 4]$. Therefore the volume of the solid of revolution is computed using the formula $V = \int_a^b \pi y^2\,dx$, which leads to

$$\begin{aligned} V &= \int_0^4 \pi(x - x^2/4)^2\,dx = \pi\int_0^4 \left(x^2 - x^3/2 + x^4/16\right)\,dx = \pi\left(x^3/3 - x^4/8 + x^5/80\right)\Big|_0^4 \\ &= \pi\left(\frac{64}{3} - 32 + \frac{64}{5}\right) = \frac{32\pi}{15}. \end{aligned}$$

Question 6 A computer scientist is modelling the spread of a computer virus on the internet. She assumes that if $y(t)$ represents the number of infected computers t days after the virus originated, then $y(t)$ can be approximated by a differentiable function satisfying

$$\frac{dy}{dt} = 0.25\, y.$$

Furthermore it is known that at time $t = 0$ there were 350 infected computers. How many computers were infected 30 days later?

Solution. This equation $\frac{dy}{dt} = 0.25y$ represents an exponential growth phenomenon, so the solution should be of the form $y(t) = Ce^{kt}$. This gives $y'(t) = k \cdot Ce^{kt} = ky$, and comparing this with the given equation we see that $k = 0.25$. The constant C represents the initial condition $C = Ce^{k0} = y(0)$, and in this case is given to be 350. It follows that $y(t) = 350e^{0.25t}$, and therefore at time $t = 30$ the number of infected computers is

$$y(30) = 350e^{0.25\times 30} = 350e^{7.5} \quad \text{(approximately equal to 632,815).}$$

Question 7 Compute the area of the two-dimensional region bounded between the y-axis, the line $y = 3$ and the curve $y = \sqrt{x}$.

Solution. By sketching the curve, it is easy to see that the area can be computed as an integral with respect to the y coordinate, as follows:

$$\text{Area} = \int_0^3 x\, dy = \int_0^3 y^2\, dy = \left.\frac{y^3}{3}\right|_0^3 = \frac{27}{3} - 0 = 9.$$

Alternatively, it can also be computed as an integral with respect to the x coordinate, but then one needs to take the difference between the straight line $y = 3$ and the curve $y = \sqrt{x}$:

$$\text{Area} = \int_0^9 (3 - \sqrt{x})\, dx = \left.\left(3x - \frac{2}{3}x^{3/2}\right)\right|_0^9 = 27 - \frac{2}{3} \cdot 27 = 9.$$

Note also that in this method the upper limit of integration is $x = 9$ which corresponds to $y = 3$ on the curve $y = \sqrt{x}$.

Question 8 Compute the center of mass of a thin rod extending between $x = 0$ and $x = 2$ with a linear mass density given by

$$\delta(x) = x + 1$$

Solution. First, compute the mass:

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (x+1)\,dx = ... = 4.$$

Next, compute the moment (about 0):

$$M_0 = \int_0^2 x\delta(x)\,dx = \int_0^2 x(x+1)\,dx = ... = \frac{14}{3}.$$

The center of mass is the ratio of these two numbers

$$\overline{x} = \frac{M_0}{M} = \frac{7}{6}.$$