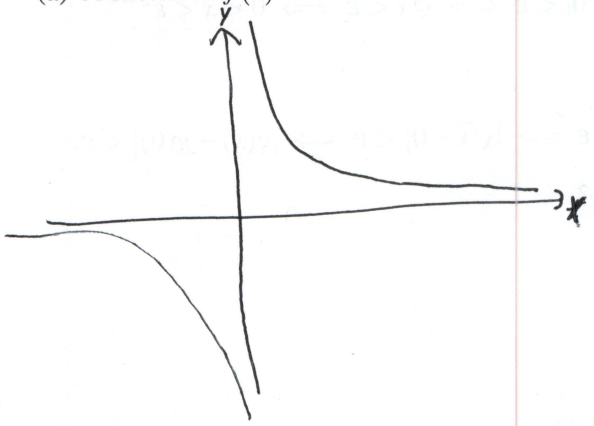
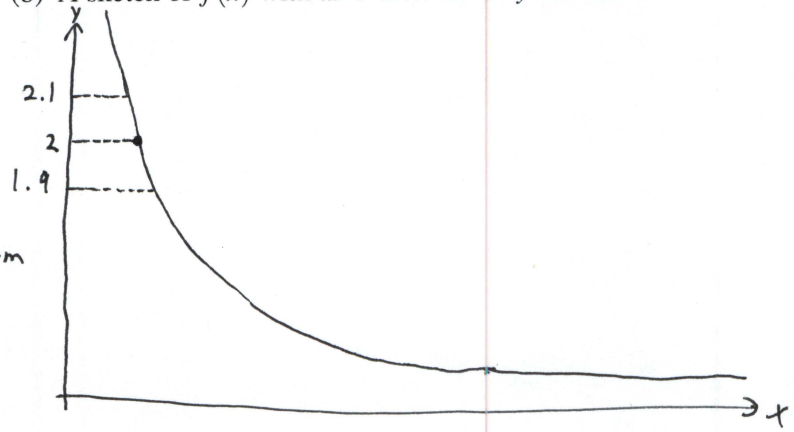
MAT 21A Homework 3: SOLUTIONS to additional problems

1. Exercise 2.3.8: We see from the graph that if x is between -3.1 and -2.9 then $f(x)$ is between 7.35 and 7.65. That is, if $|x-3|<0.1$ then $|f(x)-7.5|<\varepsilon$. So our δ is equal to 0.1. Any smaller positive number also works.
2. Exercise 2.3.10: We see from the graph that if x is between 2.61 and 3.41 then $f(x)$ is between 3.8 and 4.2. This interval is not symmetric about x, though; the biggest δ such that we can go out a distance δ in either direction from $x=3$ and still be between 2.61 and 3.41 is 0.39. So if $|x-3|<0.39$ then $|f(x)-4|<\varepsilon$. So our δ is equal to 0.39. Any smaller positive number also works.
3. $f(x)=1/x$.
   (a) A sketch of $f(x)$:

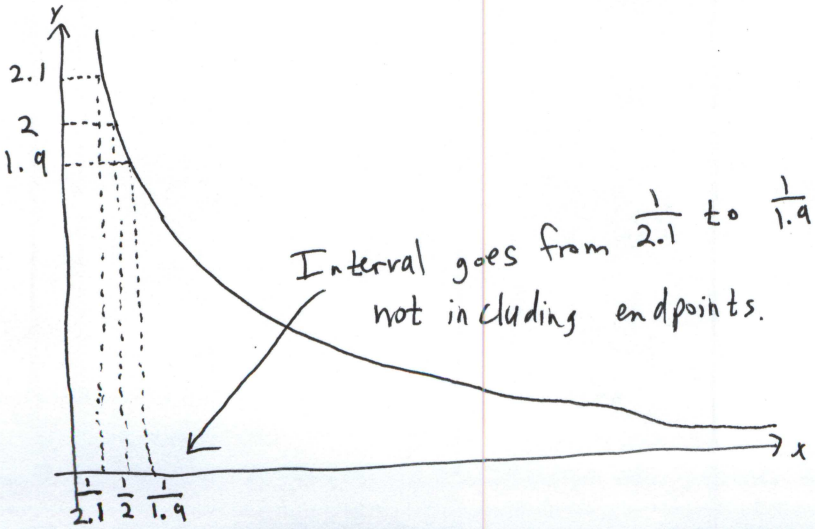


   (b) A sketch of $f(x)$ with an ε-interval for y marked:



Interval goes from 1.9 to 2.1 not including endpoints.

   (c) A sketch of $f(x)$ with ε- and δ-intervals for y and x marked:



Interval goes from $\frac{1}{2.1}$ to $\frac{1}{1.9}$ on the x-axis, not including endpoints.

4. We find expressions for δ in terms of ε to show that the limits of $f(x) = x^2$ and $g(x) = \sqrt{x}$ as $x \to 0$ are both 0:
Suppose $f(x) = x^2$.

$$|f(x) - f(0)| < \varepsilon \iff |x^2 - 0| < \varepsilon \iff x^2 < \varepsilon \iff |x| < \sqrt{\varepsilon} \iff -\sqrt{\varepsilon} < |x| < \sqrt{\varepsilon}$$

So if $\delta = \sqrt{\varepsilon}$, we have

$$|x - 0| < \delta \implies |x| < \sqrt{\varepsilon} \implies x^2 < \varepsilon \implies |x^2 - 0| < \varepsilon \implies |f(x) - f(0)| < \varepsilon.$$

So for $f(x) = x^2$ at $x = 0$, we should use $\delta = \sqrt{\varepsilon}$.
Similarly, suppose $g(x) = \sqrt{x}$.

$$|g(x) - g(0)| < \varepsilon \iff |\sqrt{x} - 0| < \varepsilon \iff \sqrt{x} < \varepsilon \iff 0 < x < \varepsilon^2$$

So if $\delta = \varepsilon^2$ and $x > 0$, we have

$$|x - 0| < \delta \implies x < \varepsilon^2 \implies \sqrt{x} < \varepsilon \implies |\sqrt{x} - 0| < \varepsilon \implies |g(x) - g(0)| < \varepsilon.$$

So for $g(x) = x^2$ at $x = 0$, we should use $\delta = \varepsilon^2$.