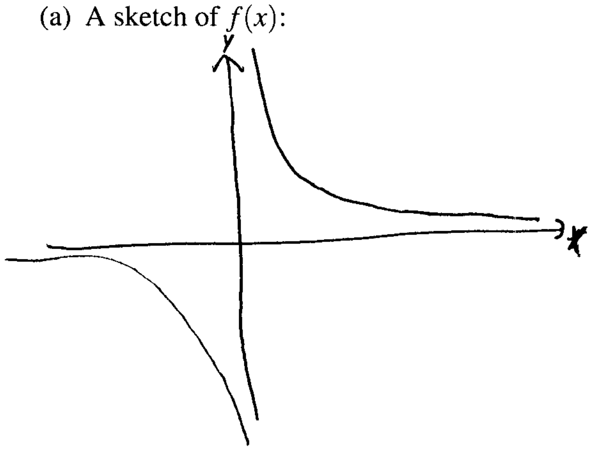
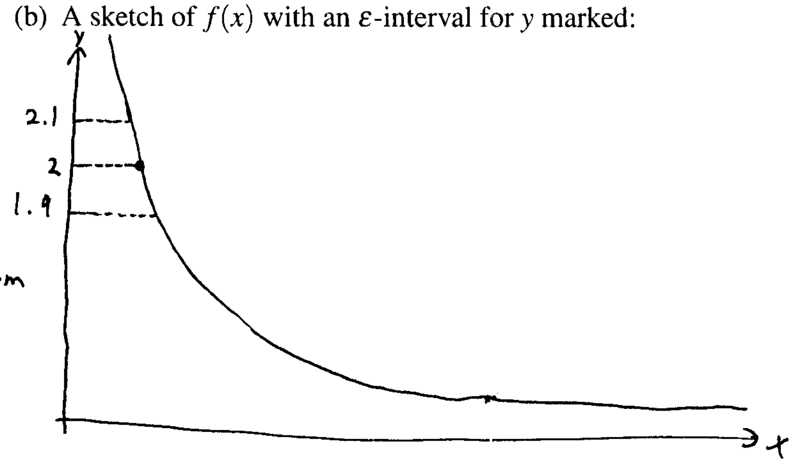
# MAT 21A Homework 3: SOLUTIONS to additional problems

1. Exercise 2.3.8: We see from the graph that if $x$ is between $-3.1$ and $-2.9$ then $f(x)$ is between 7.35 and 7.65. That is, if $|x-3| < 0.1$ then $|f(x)-7.5| < \varepsilon$. So our $\delta$ is equal to 0.1. Any smaller positive number also works.
2. Exercise 2.3.10: We see from the graph that if $x$ is between 2.61 and 3.41 then $f(x)$ is between 3.8 and 4.2. This interval is not symmetric about $x$, though; the biggest $\delta$ such that we can go out a distance $\delta$ in either direction from $x = 3$ and still be between 2.61 and 3.41 is 0.39. So if $|x-3| < 0.39$ then $|f(x)-4| < \varepsilon$. So our $\delta$ is equal to 0.39. Any smaller positive number also works.
3. $f(x) = 1/x$.
   (a) A sketch of $f(x)$:

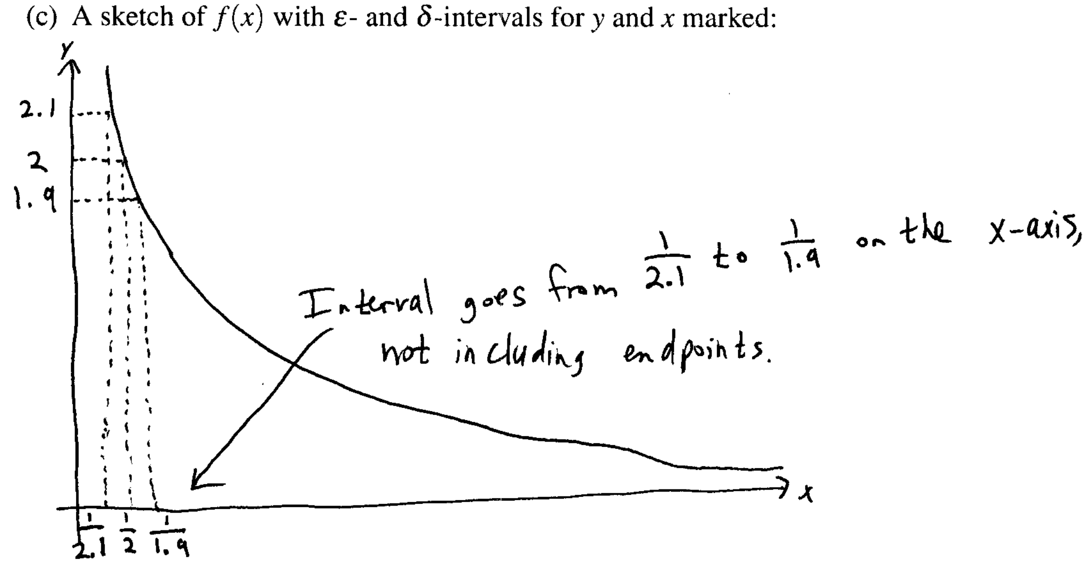



   (b) A sketch of $f(x)$ with an $\varepsilon$-interval for $y$ marked:




Interval goes from 1.9 to 2.1 not including endpoints.

   (c) A sketch of $f(x)$ with $\varepsilon$- and $\delta$-intervals for $y$ and $x$ marked:



4. We find expressions for $\delta$ in terms of $\varepsilon$ to show that the limits of $f(x) = x^2$ and $g(x) = \sqrt{x}$ as $x \to 0$ are both 0:

Suppose $f(x) = x^2$.

$$|f(x) - f(0)| < \varepsilon \iff |x^2 - 0| < \varepsilon \iff x^2 < \varepsilon \iff |x| < \sqrt{\varepsilon} \iff -\sqrt{\varepsilon} < |x| < \sqrt{\varepsilon}$$

So if $\delta = \sqrt{\varepsilon}$, we have

$$|x - 0| < \delta \implies |x| < \sqrt{\varepsilon} \implies x^2 < \varepsilon \implies |x^2 - 0| < \varepsilon \implies |f(x) - f(0)| < \varepsilon.$$

So for $f(x) = x^2$ at $x = 0$, we should use $\delta = \sqrt{\varepsilon}$.

Similarly, suppose $g(x) = \sqrt{x}$.

$$|g(x) - g(0)| < \varepsilon \iff |\sqrt{x} - 0| < \varepsilon \iff \sqrt{x} < \varepsilon \iff 0 < x < \varepsilon^2$$

So if $\delta = \varepsilon^2$ and $x > 0$, we have

$$|x - 0| < \delta \implies x < \varepsilon^2 \implies \sqrt{x} < \varepsilon \implies |\sqrt{x} - 0| < \varepsilon \implies |g(x) - g(0)| < \varepsilon.$$

So for $g(x) = x^2$ at $x = 0$, we should use $\delta = \varepsilon^2$.