1. Expander Graphs: Definitions

1.1. **Expanders: The Definition.**

Let X be a (finite) graph with edge set E and vertex set V. Let $A \subset V$. I deviate slightly from Lubotzky's notation for uniformity. We define:

- The boundary (vertex boundary) $\partial_V(A) = \{v \in V | d(v, A) = 1\}$ (where d denotes the standard metric on a graph).
- The edge boundary $\partial_E(A) = \{e \in E | e \text{ connects a vertex in } A \text{ to a vertex in } X \setminus A\}$

The literature is concerned with certain expansion ratios, which come in vertex-expansion and edge-expansion varieties.

Definition 1.1. *A graph $X = (V, E)$ is an (n, k, c)-expander if X is k-regular, $|X| = n$, and for any subset $A \subset V$, we have*

$$|\partial_V(A)| \geq c\left(1 - \frac{|A|}{n}\right)|A|$$

We can make any k-regular graph an (n, k, c)-expander for some $c > 0$ in a very trivial way (e.g. look at all subsets $A, |A| \leq n/2$). We will be interested in families of expanders with fixed k and c, and allowing $n \to \infty$.

1.2. **Bipartite Expanders.**

There is a special definition of expander when considering k-regular bipartite graphs. *A priori* we do not know that the two definitions are equivalent, and in fact they are not. However there are constructions that allow us to switch rather freely from one definition to another.

Recall that a graph X is bipartite if its vertex set can be partitioned into two subsets I, O (input and output) such that each edge has exactly one vertex in I and one vertex in O.

Definition 1.2. *Let X be a bipartite graph. X is an (n, k, c')-bipartite expander if for any $A \subset I$ with $|I| \leq n/2$, we have*

$$|\partial_A| \geq (1 + c')|A|$$

The equivalence of the two definitions is this: Given any (n, k, c)-expander X in the sense of 1.1, we can construct a bipartite (n', k', c')-expander in the sense of 1.2. Similarly the other direction. We outline the construction here.

Suppose we have X an (n, k, c) expander. We construct a bipartite graph X' in the following way:

- $I \cong O \cong V$, i.e. the input/output vertices of X' are copies of V.
- Let $v \in I, w \in O$. Then $e = (v, w)$ is an edge in X' if (v, w) is an edge in X or if $v = w$ (notation is weird; I'm labeling the vertices in I and O with the same labels as V).

Conversely, let (X, I, O) be an (n, k, c') bipartite expander.

Let $S = \{S_i\}$ be a collection of finite sets.

Definition 1.3. *A system of distinct representatives for S is a set $X = \{x_i | x_i \in S_i, x_i \neq x_j, i \neq j\}$.*

There are examples of collections which don't have systems of distinct representatives. Take $S = \mathcal{P}(\{1, 2, \ldots, n\})$ (the power set), for example. The Hall Marriage Theorem gives a condition for a collection of finite sets $S = \{S_i\}$ to have a system of distinct representatives.

Definition 1.4. *Let $S = \{S_i\}$ be a collection of finite sets. S satisfies the marriage condition if, for any collection $\{S_1, ..., S_n\} \subset S$ we have $n \leq |\bigcup S_i|$.*

This means S satisfies the marriage condition if any collection of n subsets has at least n distinct elements. The example above of $S = \mathcal{P}(\{1, \ldots, n\})$ satisfies this. In fact any subset of $\mathcal{P}$ with more than n elements fails the marriage condition. Hall's marriage theorem says this is necessary and sufficient for there to be a system of

Theorem 1.5 (Hall's Marriage Theorem)**.** *A collection S of finite sets has a system of distinct representatives if and only if S satisfies the marriage condition.*

Hall's marriage theorem applies to bipartite graphs. Recall a bipartite graph is a graph (I, O, E) where each edge is connected to one vertex in I and one vertex in O. We define a matching on a bipartite graph.

Definition 1.6. *A matching on a (bipartite) graph is a collection of non-adjacent edges. A perfect matching is a matching that covers all vertices in I.*

In our case $|I| = |O| = n$, so a perfect matching connects each vertex in I to exactly one vertex in O, hence actually covers all the vertices of X.

Applying Hall's Marriage Theorem to the bipartite graph X has a perfect matching iff $|T| \leq |N(T)|$ for all $T \subset I$ (In this case, the finite sets are $N(v)$ with $v \in S$; the representative set is the choice of neighbor $n(v)$ for $v \in S$). This is true for k-regular bipartite graph. Consider the subgraph $(T, N(T), E)$ (edges only in $T, N(T)$). There are $k|T|$ edges; if $|T| > |N(T)|$, then some vertices in $N(T)$ would have more than k incident edges, contradicting k-regularity (Pigeon Hole principle).

2. Cheeger constant and Expanders

Show: X k-regular with n vertices, then X is an $(n, k, h(X)/k)$ expander.

First assume $|A| \leq n/2$. We use the fact that $k|\partial_V A| \geq |\partial_E A|$.

$$\begin{aligned}
\frac{h(X)}{k}(1 - |A|/n)|A| &\leq \frac{|\partial_E A|}{|A|k}(1 - |A|/n)|A| \\
&\leq \frac{|\partial_V A|}{|A|}(1 - |A|/n)|A| \\
&= |\partial_V A|(1 - |A|/n) \leq |\partial_V A|
\end{aligned}$$

If $A \geq n/2$ we get

$$\begin{aligned}
\frac{h(X)}{k}(1 - |A|/n)|A| &\leq \frac{|\partial_E A|}{(n - |A|)k}(1 - |A|/n)|A| \\
&\leq \frac{|\partial_V A|}{(n - |A|)} \frac{(n - |A|)}{n}|A| \\
&= |\partial_V A| \frac{|A|}{n} \leq |\partial_V A|
\end{aligned}$$

3. Lubotzky's Proof of existence of expanders

We use a rough probabilistic estimate to show that "most" k-regular bi-partite graphs are 1/2-expanders. It is known (ref?) that any k-regular bi-partite graph with $|I| = |O| = n$ can be constructed by taking k permutations in S_n and adding an edge to vertices $(i, \pi(i))$. Therefore we identify such a bipartite graph with an n-tuple of permutations $\pi \in (S_n)^k$, and observe that $|(S_n)^k| = (n!)^k$. Note that we are overcounting the number of graphs here, but this is only a rough estimate anyway so we don't care.

Fix n and k, and let $\pi \in (S_n)^k$ be a permutation which does NOT give a 1/2-expander; call such a permutation bad. Then there exists a set $T \subset I$ and $S \subset O$, with $|S| = r \leq n/2$ and $|S| = s = 3/2r$ such that each edge e with initial vertex in T has terminal vertex in S. First fix such T, S; there are $(s(s-1)\cdots(s-r+1)(n-r)!)^k$ such permutations in $(S_n)^k$. The number of bad permutations

$$\beta \leq \sum_{r \leq n/2} \binom{n}{r}\binom{n}{s}\left(\frac{s!(n-r)!}{(s-r)!}\right)^k$$

(The following is from Michal Kapovich) Let $r \leq n/2$ and $s = [3r/2]$, $s' := [3(r+1)/2]$. Note that either

(a) $s = 3r/2$, $s' = s+1$, or

(b) $s = 3r/2 + 1/2$ and $s' = s+2$.

We are trying to estimate

$$\sum_{r=1}^{[n/2]} B(r),$$

where

$$B(r) = \binom{n}{r}\binom{n}{s}(\binom{s}{r}(n-r)!)^k.$$

Then

$$\frac{B(r+1)}{B(r)} = R_1 R_2 (R_3 R_4)^k$$

where

$$B_1 = (\binom{n}{r+1})\binom{n}{r}^{-1} = \frac{n-r}{r+1} < n,$$

$$R_2 = (\binom{n}{s'})(\binom{n}{s})^{-1} \leq \frac{(n-s)!}{(n-s-2)!} \leq n^2,$$

$$R_3 = (\binom{s'}{r+1})(\binom{s}{r})^{-1}$$

In case (a) we obtain $R_3 \leq 3/2$, in case (b) we obtain $R_3 \leq 10$. In any case, $R_3 \leq 10$.

$$R_4 = \frac{(n-r-1)!}{(n-r)!} = \frac{1}{n-r} \leq \frac{2}{n}.$$

Putting all this together, we get

$$\frac{B(r+1)}{B(r)} \leq n^3(\frac{20}{n})^k \leq 20^3 \cdot (\frac{20}{n})^{k-3}.$$

Hence, for $k \geq 4$, $n \geq 20^4$ we obtain

$$\frac{B(r+1)}{B(r)} \leq 1.$$

Therefore,

$$S = \sum_{r=1}^{[n/2]} B(r) \leq nB(1) \leq n^3(2(n-1)!)^k < (n!)^k.$$

Moreover, we can make the ratio $S/(n!)^k$ as small as possible by taking sufficiently large n.