IKFG Lecture about the laplacian

★ We are interested in the discrete Laplace operator, an analog to the continuous Laplace operator, defined so that it has meaning on a graph.

★ we have the usual defn. of the Laplacian on a manifold:

$\Delta = \operatorname{div} \circ \operatorname{grad}$ (divergence of the gradient)

If we have time at the end, I can give the defn. of div and grad in terms of a graph.

★ defn: the Laplacian of a graph $G=(V,E)$ is $\Delta = D - A$, where A is the adjacency matrix of G and D is the diagonal matrix with entries $D_{ii} = \deg i$.

Note: if G is k-regular, $\Delta = kI - A$.

★ Exercise 3.2.5: (from Linear Algebra notes)

If G is a k-regular graph and $k = \mu_1 \geq \dots \geq \mu_n$ are eigenvalues of A and $0 = \lambda_1 \leq \dots \leq \lambda_n$ are eigenvalues of Δ, then $\lambda_i + \mu_i = k$. In particular, $\lambda_2 = \mu_1 - \mu_2$.

Note: λ_2 is the spectral gap.

Proof: Consider the characteristic polynomial $p(x) = |A - xI|$. Then $(x-\mu_1)(x-\mu_2)\cdots(x-\mu_n)$ are its factors.

Now, $|\Delta - \lambda I| = |D - A - \lambda I| = |-A + B|$, where

$$B = D - \lambda I = \begin{bmatrix} \deg 1 - \lambda & & \\ & \ddots & \\ & & \deg n - \lambda \end{bmatrix} = \begin{bmatrix} k-\lambda & & 0 \\ & \ddots & \\ 0 & & k-\lambda \end{bmatrix}$$

So that $|-A+B| = |-1(A-B)| = (-1)^n |A-B| = (-1)^n |A - (k-\lambda)I|$.

Let $y = k - \lambda$. Then $p(y) = (y-\mu_1)(y-\mu_2)\cdots(y-\mu_n)$ and

$|\Delta - \lambda I| = (-1)^n \prod_i (k - \lambda - \mu_i) = 0 \implies \lambda_i = k - \mu_i$, as desired.

Since $\mu_1 = k$, it is clear that $\lambda_2 = k - \mu_2 = \mu_1 - \mu_2$. ∎

★ Now I state the Thm. that Shinpei gave last week, but this time its in terms of the Laplacian.

Thm. Let X be a finite, connected, k-regular graph with $0 = \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$ the eigenvalues of Δ. Then

$$0 < \frac{\lambda_2}{2} \le h(X) \le \sqrt{2k \cdot \lambda_2}$$

[this is because $\frac{k-\mu_2}{2} = \frac{\lambda_2}{2}$ and $k - \mu_2 = \mu_1 - \mu_2$ and $\mu_1 \ne \mu_2$ iff X is connected $\Rightarrow \lambda_2 \ne 0$ iff X is connected]

★ **Recall**: The Cheeger constant of a graph X is

$$h(X) = \inf_{\substack{F \subseteq V \\ |F| \le \frac{|V|}{2}}} \frac{|\partial F|}{|F|}$$

EX:

for C_n: an upper bound on $h(C_n)$ is $\frac{2}{\lfloor \frac{n}{2} \rfloor}$ (found by considering a chain of $\lfloor \frac{n}{2} \rfloor$ vertices)

Hence $h(C_n) \to 0$ as $n \to \infty$.
Such a network is undesireable for practical purposes

for K_n: $h(K_n) = \lceil \frac{n}{2} \rceil$, much more desireable than C_n.

★ Now let's talk about families of expanders.

defn. Let $(X_m)_{m \ge 1}$ be a family of graphs $X_m = (V_m, E_m)$, $m \in \mathbb{N}$. Fix $k \ge 2$. Such a family $(X_m)_{m\ge1}$ of finite, connected, k-regular, graphs is a family of expanders if:

(1) $|V_m| \to \infty$ as $m \to \infty$

(2) $\exists\ \varepsilon > 0$ s.t. $h(X_m) \ge \varepsilon \quad \forall\ m \ge 1$.

$\longrightarrow$

Note: the condition that X_m is k-regular assures that $|E_m|$ grows linearly w/ $|V_m|$.

Examples:

① family of 8-regular graphs G_m $\forall\, m \in \mathbb{Z}$.

Here $V_m = \mathbb{Z}_m \times \mathbb{Z}_m$ and the neighbors of the vertex (x,y) are $(x+y, y)$, $(x-y, y)$, $(x, y+x)$, $(x, y-x)$, $(x+y+1, y)$, $(x-y+1, y)$, $(x, y+x+1)$, $(x, y-x+1)$.

(all operations are mod m).

② family of 3-regular p-vertex graphs $\forall$ prime p.

- $V_p = \mathbb{Z}_p$
- a vertex x is connected to $x+1$, $x-1$ & x^{-1}
  - all operators are mod p
  - define 0^{-1} to be 0

defn. a Ramanujan graph G is a regular graph of degree r s.t. $\forall\, i \geq 2$, $|\lambda_i| \leq \sqrt{2r-1}$

OR: a finite, connected, k-regular graph X is Ramanujan if, $\forall$ eigenvalue μ of A other than $\pm k$, one has $|\mu| \leq 2\sqrt{k-1}$.

extra :

Analogous defn. for div & grad of graphs :

Let G be a graph

Let $K_{(v,e)} = \begin{cases} 1 & \text{if } e = (v,*) \\ -1 & \text{if } e = (*,v) \\ 0 & \text{else} \end{cases}$

define :

grad: $L^2VG \longrightarrow L^2EG$

given by $(f : V \to \mathbb{R}) \longmapsto fK$

where $(fK)_{e=(u,v)} = f_u - f_v$

div : $L^2EG \longrightarrow L^2VG$

given by $(g : E \longrightarrow \mathbb{R}) \longmapsto Kg$

where $Kg = \sum_{e=(v,*)} g_e - \sum_{e=(*,v)} g_e$