

The Haagerup Property

- a generalization of amenability & strong negation of Property (T)

Assume G second-countable & locally compact.

Property (T)

- $\forall \pi$ with $1_G \in \pi$, $1_G \in \pi$
- \forall continuous isometric action $G \curvearrowright H$, π has a fixed point

proper: \forall bounded $B \subset H$,
Eg: $\{B \cap \mathbb{R}\}$ is rel. compact
normalized: $\phi_n(x) = \dots$

let (ϕ_n) denote a sequence of continuous normalized positive definite functions on G converging to 1 uniformly on compact sets in G

- $\forall (\phi_n)$, ϕ_n converges uniformly to 1 on G
- \forall conditionally negative definite $\Psi: G \rightarrow \mathbb{R}^+$
 Ψ is bounded

Haagerup property

$\lim_{g \rightarrow \infty} \pi(g) \psi \rightarrow 0$

think left regular

- $\exists \pi$ with $1_G \notin \pi$ whose coefficients vanish at infinity
- \exists continuous isometric action $G \curvearrowright H$ which is proper

Admits a metrically proper isometric action on $H^{\infty}(\mathbb{R})$ or $H^{\infty}(\mathbb{C})$

curvature

- Passes to \dots subgroups, ^{finite} direct products, free products, amalgamated products, semi-direct products $\mathbb{R}^2 \rtimes \mathbb{Z}$, $\mathbb{R}^2 \rtimes \mathbb{Z}_2$, $\mathbb{R}^2 \rtimes \mathbb{R}$

Examples:

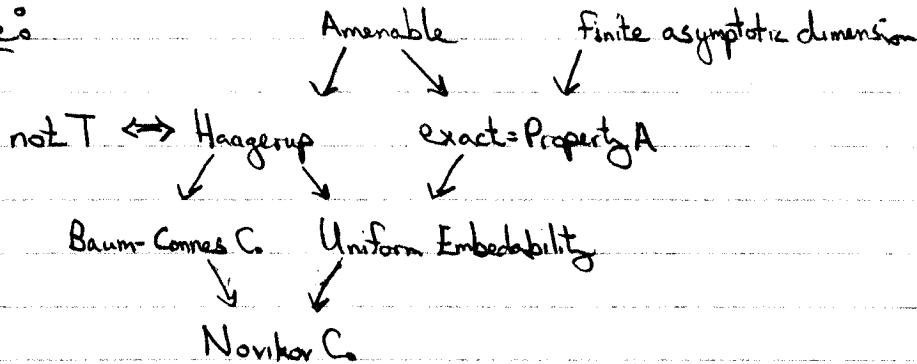
- Compact groups (trivially)
 - Compact \iff ^{Haagerup} amenable + property (T)
- Amenable groups ($1_G \in L$)
- The Lie groups $SO(n,1)$ and $SU(n,1)$
 - actions are either proper or have a fixed point
 - $g \mapsto d(gx_0, x_0)$ is cond. neg. def.
- Groups acting properly on a ^{tree} T with ^{finite} edge stab & Haagerup vertex stab
 - countably gen free ^{grps} (word length is cond. neg. def) \leftarrow note not amenable
 - $g \mapsto d(gx_0, x_0)$ again cond. neg. def on $\text{Aut}(T)$
- Coxeter groups
- Groups acting properly on a ^{space with walls} $AT(\mathbb{C})$ cubical complex
 - $g \mapsto d(gx_0, x_0)$ and
 - contains Thompson's F
- bounded automatic automorphisms of a rooted tree

tree

walls partition space with walls: finitely many walls separate any two points proper action preserves \mathcal{W} , $g \mapsto d(gx_0, x_0)$ proper

Theorem: (strong rigidity): Let Γ be a discrete group with the Haagerup property. There is a 1-parameter family of unitary representations of Γ that "interpolates" between the trivial and left regular representations.

Big Picture



Theorem: Let G be a connected Lie group. The following are equivalent:

- G has the Haagerup property
- For every closed subgroup $H < G$, if (G, H) has relative property (T), then H is compact. known for other gps?
- G is locally isometric to $M \times SO(n_1, 1) \times \dots \times SO(n_r, 1) \times SU(m_1, 1) \times \dots \times SU(m_s, 1)$ where M is an amenable Lie group

compact ext. of a sol. Lie gr

Open questions

- Is Haagerup geometric? ^{ig. qis inv.} Not property (T).
- Other obstructions to Haagerup?
- On relativ. gps? (known for $B(\mathbb{R}^n)$) 3-manifold gps? Braid gps? (known for $n=3$)

Definition: A f.g. group Γ has property A if for some finite generating set S , for all $\epsilon > 0$, there exists finite subsets $A_\epsilon \subset \Gamma \times \mathbb{N}$.

- (1) $(\gamma, i) \in A_\epsilon$
- (2) $|A_\epsilon \Delta A_\epsilon| < \epsilon |A_\epsilon \cap A_\epsilon|$ whenever $|\gamma^{-1}\delta|_S \leq \epsilon$
- (3) $\exists R > 0$: $(x, m), (y, n) \in A_\epsilon \Rightarrow |x^{-1}y|_S \leq R$

Definition: A f.g. group Γ admits a uniform embedding into Hilbert space if there exists $\beta: \Gamma \rightarrow H$ + $\rho_1, \rho_2: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ nondecreasing s.t.

- (1) $\lim_{r \rightarrow \infty} \rho_1(r) = \infty$
- (2) $\rho_1(\|x^{-1}y\|) \leq \|\beta(x) - \beta(y)\| \leq \rho_2(\|x^{-1}y\|)$ for all $x, y \in \Gamma$