28Feb08
Pg 1

GGT RFG on exanders & amenability

The Haagerup Property

- a generalization of amenability & strong negation of Property (T)

Assume G second-countable & locally compact.

Property (T)	Haagerup property $\lim_{g\to\infty}\langle \pi g v, v\rangle$ think left regular
$\forall\ \pi$ with $1_G \prec \pi$, $1_G \subset \pi$	$\exists\ \pi$ with $1_G \prec \pi$ whose coefficients vanish at infinity
$\forall$ continuous isometric action $G \overset{\alpha}{\curvearrowright} H$, α has a fixed point	$\exists$ continuous isometric action $G \overset{\alpha}{\curvearrowright} H$ which is proper

proper: $\forall$ bounded $B \subset H$, $\{g \mid gB \cap B \neq \emptyset\}$ is rel. compact

normalized: $\phi_n(1)=1$

Let (ϕ_n) denote a sequence of continuous normalized positive definite functions on G converging to 1 uniformly on compact sets in G

$\forall\ (\phi_n)$, ϕ_n converges uniformly to 1 on G	$\exists (\phi_n)$ which vanishes at infinity
$\forall$ conditionally negative definite $\Psi: G \to \mathbb{R}^+$, Ψ is bounded	$\exists$ conditionally negative definite $\Psi: G \to \mathbb{R}^+$ which is proper
	Admits a metrically proper isometric action on $\mathbb{H}^\infty(\mathbb{R})$ or $\mathbb{H}^\infty(\mathbb{C})$

curvature

- Passes to ... subgroups, finite direct products, free products, amalgamated products. semi-direct products? $\mathbb{R}^2 \rtimes SL_2\mathbb{R}$: no.

Examples:

- Compact groups (trivially)
  - Compact $\iff$ Haagerup ~~amenable~~ + property (T)
- Amenable groups ($1_G \prec L$)
- The Lie groups $SO(n,1)$ and $SU(n,1)$
  - actions are either proper or have a fixed point
  - $g \mapsto d(gx_0, x_0)$ is cond. neg. def.
- Groups acting properly on a tree like T with finite edge stab & Haagerup vertex stab (can generalize?)
  - countably gen free grps (word length is cond neg def) ← note not amenable
  - $g \mapsto d(gx_0, x_0)$ again cond. neg. def on $Aut(T)$
- Coxeter groups
- Groups acting properly on a space with walls CAT(0) cubical complex
  - $g \mapsto d_W(gx_0, x_0)$ cnd
  - contains Thompson's F
- bounded automatic automorphisms of a rooted tree

wall: partition
space with walls: finitely many walls separate any two points
proper action: preserves W, $g \mapsto w(g x_0, x_0)$ is proper

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<u>Theorem</u>: (strong nonrigidity): Let Γ be a discrete group with the Haagerup property. There is a 1-parameter family of unitary representations of Γ that "interpolates" between the trivial and left regular representations.

<u>Big Picture</u>:

$$\begin{tikzcd}
 & & \text{Amenable} \arrow[dl] \arrow[dr] & & \text{finite asymptotic dimension} \arrow[dl] \\
\text{not } T \arrow[r, leftrightarrow] & \text{Haagerup} \arrow[dl] \arrow[dr] & & \text{exact} = \text{Property } A \arrow[dl] & \\
\text{Baum-Connes C.} \arrow[dr] & & \text{Uniform Embedability} \arrow[dl] & & \\
 & \text{Novikov C.} & & &
\end{tikzcd}$$

<u>Theorem</u>: Let G be a connected Lie group. The following are equivalent:
- G has the Haagerup property
- For every closed subgroup $H<G$, if (G,H) has relative property (T), then H is compact — known for other gps?
- G is locally isometric to
$$M \times SO(n_1,1)\times \dots \times SO(n_k,1)\times SU(m_1,1)\times\dots\times SU(m_\ell,1)$$
where M is an amenable Lie group — compact ext. of a solv. Lie gp

Open questions:
- Is Haagerup geometric? (i.e. q.i. inv.) Not property (T)
- Other obstructions to Haagerup?
- One relator gps? (known for $BS(p,q)$) 3-manifold gps? Braid gps? (known for $n=3$)

<u>Definition</u>: A f.g. group Γ has <u>property A</u> if for some finite generating set S, for all $r,\varepsilon>0$, there exists finite subsets $A_\gamma \subset \Gamma\times\mathbb{N}$:
- (1) $(\gamma,1)\in A_\gamma$
- (2) $|A_\gamma \Delta A_{\gamma'}| < \varepsilon |A_\gamma \cap A_{\gamma'}|$ whenever $|\gamma^{-1}\gamma'|_S \leq r$
- (3) $\exists R>0$: $(x,m),(y,n)\in A_\gamma \Rightarrow |x^{-1}y|_S \leq R$

<u>Definition</u>: A f.g. group Γ admits a uniform embedding into Hilbert space if there exists $\beta:\Gamma\to H$ + $\rho_1,\rho_2:\mathbb{R}^+\to\mathbb{R}^+$ nondecreasing s.t.:
- (1) $\lim_{r\to\infty}\rho_i(r)=\infty$
- (2) $\rho_1(|x^{-1}y|)\leq \|\beta(x)-\beta(y)\|\leq \rho_2(|x^{-1}y|)$ for all $x,y\in\Gamma$