MAT 121: Advanced Analysis for the Sciences
Final Exam Answers

Problem 1 (20 pts) Consider the following heat equation on a 1D rod:

$$u_t = \alpha^2 u_{xx}, \quad t \geq 0,\ -\infty < x < \infty,$$

with the initial condition $u(x, 0) = f(x)$.

(a) (10 pts) Solve this PDE. [Hint: Use the Fourier transform. Which variable do you want to apply the Fourier transform, x or t? You may also want to use some formula in the Fourier transform table.]

Answer: We apply the Fourier transform in x on both sides of these equations. Assuming that u and f are in $L^1(\mathbb{R})$ (i.e., can apply the Fourier transform), we get

$$\frac{\partial \widehat{u}}{\partial t}(\xi, t) = \alpha^2 (\mathrm{i}\xi)^2 \widehat{u}(\xi, t) = -\alpha^2 \xi^2 \widehat{u}(\xi, t) \tag{1}$$

with

$$\widehat{u}(\xi, 0) = \widehat{f}(\xi). \tag{2}$$

Now, for each fixed ξ, (1) can be viewed as a simple ODE in t, with the initial condition (2). Therefore, we can easily solve this to get

$$\widehat{u}(\xi, t) = \widehat{f}(\xi) \mathrm{e}^{-\alpha^2 \xi^2 t}.$$

Now, we apply the inverse Fourier transform in ξ to get the solution:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{u}(\xi, t) \mathrm{e}^{\mathrm{i}\xi x} \mathrm{d}\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) \mathrm{e}^{-\alpha^2 \xi^2 t} \mathrm{e}^{\mathrm{i}\xi x} \mathrm{d}\xi = \mathcal{F}^{-1}\left[\widehat{f}(\xi) \mathrm{e}^{-\alpha^2 t \xi^2}\right]. \tag{3}$$

So, this is the inverse Fourier transform of the product of two functions. Here we can use the convolution formula

$$\mathcal{F}[g \star h] = \widehat{g}\,\widehat{h} \Longrightarrow g \star h = \mathcal{F}^{-1}\left[\widehat{g}\,\widehat{h}\right].$$

Now, the inverse Fourier transform of $\widehat{f}(\xi)$ is of course $f(x)$, so, (3) is now written as:

$$u(x, t) = f \star \mathcal{F}^{-1}\left[\mathrm{e}^{-\alpha^2 t \xi^2}\right].$$

To compute the inverse Fourier transform of $\mathrm{e}^{-\alpha^2 t \xi^2}$, we simply refer the provided table:

$$\exp\left(-\frac{a x^2}{2}\right) \xrightarrow{\mathcal{F}} \sqrt{\frac{2\pi}{a}} \exp\left(-\frac{\xi^2}{2a}\right)$$

So, by setting $1/(2a) = \alpha^2 t$, i.e., $a = 1/(2\alpha^2 t)$, we have

$$\mathcal{F}^{-1}\left[\mathrm{e}^{-\alpha^2 t \xi^2}\right] = \frac{1}{\sqrt{4\pi\alpha^2 t}} \mathrm{e}^{-x^2/(4\alpha^2 t)}.$$

Plugging this into (3) and using the definition of convolution, finally we have:

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} \int_{-\infty}^{\infty} \mathrm{e}^{-(x-y)^2/(4\alpha^2 t)} f(y) \mathrm{d}y.$$

(b) (10 pts) Suppose the domain is changed to $0 < x < \infty$. And suppose initially the rod is in steady state with $u(x,t) = 0$, for $t < 0$, and $x \in [0, \infty)$. Now at $t = 0$ and on, we set the boundary condition as $u(0,t) = T_0$ for $t \geq 0$. Solve this heat equation. [Hint: Use the Laplace transform. Which variable do you want to apply the Fourier transform, x or t? You may also want to use some formula in the Laplace transform table.]

Answer: In this case, we take the Laplace transform in t of both sides of the heat equation. Let $U(x,p)$ be the Laplace transform of $u(x,t)$, i.e.,

$$U(x,p) = \int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{-pt}\mathrm{d}t.$$

Then, the heat equation becomes

$$pU - u(x,0) = pU = \alpha^2 \frac{\partial^2 U}{\partial x^2},$$

since $u(x,0) = 0$. Viewing p as a constant, this is a 2nd order ODE in terms of x. So, we can easily solve this as:

$$U(x,p) = A\mathrm{e}^{(\sqrt{p}/\alpha)x} + B\mathrm{e}^{-(\sqrt{p}/\alpha)x},$$

where A, B are some constants. But, to prevent the blowup as $x \to \infty$, we must have $A = 0$. To determine B, we use the initial condition, $u(0,t) = T_0$. By applying the Laplace transform of both sides of this initial condition gives us:

$$U(0,p) = \frac{T_0}{p}.$$

So, $B = T_0/p$. In other words,

$$U(x,p) = \frac{T_0}{p}\mathrm{e}^{-(\sqrt{p}/\alpha)x}.$$

From the Laplace transform table,

$$(1/p)\mathrm{e}^{-a\sqrt{p}} \xrightarrow{\mathcal{L}^{-1}} 1 - \mathrm{erf}(a/(2\sqrt{t})).$$

So, by replacing a by x/α, we have:

$$u(x,t) = T_0\left[1 - \mathrm{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)\right].$$

Problem 2 (10 pts) Solve the following ODE using the Laplace transform:

$$y'' + 4y = H(t-\pi),$$

with $y(0) = y'(0) = 0$. Note that $H(\cdot)$ is the Heaviside step function.

Answer: Apply the Laplace transform to the both sides. Let $Y(p)$ is the Laplace transform of $y(t)$. Since $y(0) = y'(0) = 0$, we get:

$$(p^2+4)Y(p) = \mathfrak{L}[H(t-\pi)].$$

Therefore,

$$Y(p) = \frac{1}{p^2+4} \cdot \mathfrak{L}[H(t-\pi)] = \mathfrak{L}\left[\frac{1}{2}\sin 2t\right] \cdot \mathfrak{L}[H(t-\pi)].$$

Now, we use the convolution formula:

$$F(p)G(p) \xrightarrow{\mathfrak{L}^{-1}} (f \star g)(t).$$

So,

$$\begin{aligned} y(t) &= \frac{1}{2}\sin 2t \star H(t-\pi) \\ &= \frac{1}{2}\int_0^t \sin 2(t-\tau)H(\tau-\pi)\mathrm{d}\tau \end{aligned}$$

Now,

$$H(\tau-\pi) = \begin{cases} 0 & \text{if } \tau < \pi; \\ 1 & \text{if } \tau > \pi. \end{cases}$$

So, if $t > \pi$, then

$$y(t) = \frac{1}{2}\int_\pi^t \sin 2(t-\tau)\mathrm{d}\tau = \frac{1}{4}(1-\cos 2t),$$

while if $t < \pi$, then clearly $y(t) = 0$. So, we have:

$$y(t) = \frac{1}{4}H(t-\pi)(1-\cos 2t).$$

Problem 3 (10 pts) Solve the following ODE using Green's function:

$$y'' + 4y = f(t),$$

with $y(0) = y'(0) = 0$ and $f(t)$ is a forcing function defined for $t \geq 0$.

Answer: Let Green's function of this problem be $G(t,\tau)$. Then, this function satisfies the following ODE:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}G(t,\tau) + 4G(t,\tau) = \delta(t-\tau),$$

where $\delta(\cdot)$ is the Dirac delta function. We need to solve this ODE with the initial condition $G(0,\tau) = \frac{\mathrm{d}}{\mathrm{d}t}G(0,\tau) = 0$. So, the simplest way to solve this is to use the Laplace transform. Let $\tilde{G}(p,\tau)$ be the Laplace transform of $G(t,\tau)$. Then, we get

$$(p^2+4)\tilde{G}(p,\tau) = \mathfrak{L}[\delta(t-\tau)],$$

i.e.,

$$\tilde{G}(p,\tau) = \frac{1}{p^2+4}\cdot\mathfrak{L}[\delta(t-\tau)].$$

Using the convolution formula similarly to the argument in Problem 2, we have

$$\begin{aligned} G(t,\tau) &= \frac{1}{2}\int_0^t \sin 2(t-\eta)\delta(\eta-\tau)\mathrm{d}\eta \\ &= \frac{1}{2}\int_{-\tau}^{t-\tau} \sin 2(t-\tau-\omega)\delta(\omega)\mathrm{d}\omega \quad \text{by change of variable } \omega = \eta - \tau. \\ &= \begin{cases} \frac{1}{2}\sin 2(t-\tau) & \text{if } 0 < \tau < t; \\ 0 & \text{if } 0 < t < \tau. \end{cases} \end{aligned}$$

Therefore, finally, we have the following solution:

$$y(t) = \int_0^t G(t,\tau)f(\tau)\mathrm{d}\tau = \frac{1}{2}\int_0^t \sin 2(t-\tau)f(\tau)\mathrm{d}\tau.$$

Problem 4 (10 pts) Find the best (in the least square sense) third-degree polynomial approximation to

$$f(x) = \chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } \ 0 < x < 1; \\ 0 & \text{if } \ -1 < x < 0. \end{cases}$$

Because the Legendre series expansion of $f(x)$ upto order three gives the least squares solution, it suffices to compute the four term Legendre expansion of $f(x)$, i.e., the best third order polynomial approximation in the least square sense is

$$c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x),$$

where

$$c_k = \frac{2k+1}{2}\langle f, P_k \rangle = \frac{2k+1}{2}\int_{-1}^{1} f(x) P_k(x)\mathrm{d}x.$$

In this case, $f(x) = \chi_{[0,1]}(x)$, so the integral becomes very simple:

$$c_k = \frac{2k+1}{2}\int_{0}^{1} f(x) P_k(x)\mathrm{d}x.$$

Now, we need to compute them for $k = 0, 1, 2, 3$.

$$c_0 = \frac{1}{2}\int_0^1 P_0(x)\mathrm{d}x = \frac{1}{2}\int_0^1 1\mathrm{d}x = \frac{1}{2}.$$

$$c_1 = \frac{3}{2}\int_0^1 P_1(x)\mathrm{d}x = \frac{3}{2}\int_0^1 x\mathrm{d}x = \frac{3}{4}.$$

$$c_2 = \frac{5}{2}\int_0^1 P_2(x)\mathrm{d}x = \frac{5}{2}\int_0^1 \frac{1}{2}(3x^2 - 1)\mathrm{d}x = \frac{5}{4}[x^3 - x]_0^1 = 0.$$

$$c_3 = \frac{7}{2}\int_0^1 P_3(x)\mathrm{d}x = \frac{7}{2}\int_0^1 \frac{1}{2}(5x^3 - 3x)\mathrm{d}x = \frac{7}{4}\left[\frac{5}{4}x^4 - \frac{3}{2}x^2\right]_0^1 = -\frac{7}{16}.$$

Therefore, the third-order least squares polynomial is:

$$\frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) = \frac{1}{2} + \frac{45}{32}x - \frac{35}{32}x^3.$$

Problem 5 (10 pts) Find the general solution of the following ODE using the method of Frobenius (i.e., the generalized power series):

$$2x^2y'' + 3xy' - y = 0, \quad x > 0.$$

Answer: Our strategy is to plug in $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ to this ODE, and find $\{a_n\}$ and s. Since

$$y' = \sum_n a_n(n+s)x^{n+s-1}, \quad y'' = \sum_n a_n(n+s)(n+s-1)x^{n+s-2},$$

we have the following table:

	x^s	x^{s+1}	$\dots$	x^{n+s}	$\dots$
$2x^2y''$	$2s(s-1)a_0$	$2a_1(1+s)s$	$\dots$	$2a_n(n+s)(n+s-1)$	$\dots$
$3xy'$	$3sa_0$	$3a_1(1+s)$	$\dots$	$3a_n(n+s)$	$\dots$
$-y$	$-a_0$	$-a_1$	$\dots$	$-a_n$	$\dots$

Therefore, from the x^s (i.e., a_0) term, we have:

$$a_0(2s(s-1) + 3s - 1) = 0.$$

Assuming a_0 is not zero, we have

$$(2s(s-1) + 3s - 1) = 2s^2 + s - 1 = (s+1)(2s-1) = 0.$$

Therefore $s = -1$ or $s = 1/2$.

$\underline{s = -1 \textbf{ case:}}$ Consider the $x^{n+s} = x^{n-1}$ term (i.e., the term with a_n). We have:

$$(2(n-1)(n-2) + 3(n-1) - 1)a_n = n(2n-3)a_n = 0.$$

Since n is a non-negative integer, $a_n = 0$ for $n = 1, 2, \dots$ So, we have a fundamental solution $y = x^{-1}$.

$\underline{s = 1/2 \textbf{ case:}}$ In this case, the term with a_n is:

$$(2(n+1/2)(n-1/2) + 3(n+1/2) - 1)a_n = n(2n+3)a_n = 0.$$

By the similar reasoning as above, $a_n = 0$ for $n = 1, 2, \dots$ So, we have another fundamental solution $y = x^{1/2} = \sqrt{x}$.

A general solution: is a linear combination of the fundamental solutions. Thus we have:

$$y = \frac{A}{x} + B\sqrt{x},$$

where A and B are arbitrary constants.

Problem 6 (20 pts) We want to find the steady-state temperature distribution u in a semi-infinite solid cylinder of radius 1 if the base is held at T_0 degree and the side wall at 0 degree.

(a) (5 pts) Do the separation of the above equation by assuming $u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, and derive the ODEs of R, Θ, and Z.

Answer: The Laplace equation in the cylindrical coordinate (r, θ, z) is:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Now, assume $u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, and plug this into the Laplace equation, and then divide both sides by $R\Theta Z$ to get:

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} + \frac{Z''}{Z} = 0, \tag{4}$$

with the understanding that $R'' = \dfrac{\mathrm{d}^2 R}{\mathrm{d}r^2}$, $\Theta'' = \dfrac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}$, etc. Now, the Z''/Z part is a function of z only while the other part is a function of r, θ. Therefore, Z''/Z must be a constant and let the separation constant k^2, i.e.,

$$\boxed{\frac{Z''}{Z} = k^2, \quad k > 0.} \tag{5}$$

Then, (4) becomes:

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -k^2,$$

Multiplying r^2 on both sides and moving the Θ term to the right, we have:

$$\frac{r^2R''}{R} + \frac{rR'}{R} + k^2r^2 = -\frac{\Theta''}{r^2\Theta}.$$

Now, the left-hand side is a function of r only while the right-hand side is a function of θ only. Therefore, this must be a constant, and let this separation constant be n^2. So, in terms of Θ, we have:

$$\boxed{\frac{\Theta''}{\Theta} = -n^2, \quad n > 0.} \tag{6}$$

Finally, we thus have the ODE for R as:

$$\boxed{r^2R'' + rR' + (k^2r^2 - n^2)R = 0,} \tag{7}$$

which is the so-called Bessel's equation.

(b) (5 pts) Solve the above ODEs first to get all possible solutions without considering the boundary conditions.

Answer: From (5), we get

$$\boxed{Z(z) = \begin{cases} \mathrm{e}^{kz} \\ \mathrm{e}^{-kz} \end{cases} .} \tag{8}$$

As for Θ, the basic solutions of (6) is

$$\boxed{\Theta(\theta) = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} .} \tag{9}$$

Finally, for R, (7) is Bessel's equation, so the basic solutions are

$$\boxed{R(r) = \begin{cases} J_n(kr) \\ N_n(kr) \end{cases} ,} \tag{10}$$

where $J_n(\cdot)$ and $N_n(\cdot)$ are the Bessel functions of a first and a second kind (of order n), respectively.

(c) (5 pts) Select the possible solutions of those ODEs by matching the boundary conditions and other geometric considerations.

Answer: The domain is a semi-infinite cylinder, and $u \to 0$ as $z \to \infty$. Therefore, the only possibility in (8) is

$$\boxed{Z(z) = \mathrm{e}^{-kz}.}$$

As for Θ, from the boundary condition $u(1, \theta, z) = 0$ and $u(r, \theta, 0) = T_0$, the solution does not have angular dependency, i.e., the solution should not depend on θ. Therefore, $n = 0$ is the only possibility in (9), but $\Theta(\theta)$ should not be always zero (otherwise, $u \equiv 0$). Therefore, $\cos 0 = 1$ is the only solution, i.e.,

$$\boxed{\Theta(\theta) = 1.}$$

Finally, in (10), since $N_n(kr)$ blows up at $r = 0$, this must be excluded. Moreover, from the angular independence, $n = 0$ is the only possibility. Thus, we have:

$$\boxed{R(r) = J_0(kr).}$$

(d) (5 pts) Find the final solution u by matching the boundary condition at the bottom and the cylinder wall.

Answer: Since $u(1, \theta, z) =$, we must have

$$R(1) = J_0(k) = 0.$$

Let k_m, $m = 1, 2, \ldots$, be the zeros of J_0. Linearly combining the ODE solutions derived in part (c), we thus form the following linear combination of the basic solutions:

$$u(r, \theta, z) = \sum_{m=1}^{\infty} c_m J_0(k_m r) \mathrm{e}^{-k_m z},$$

where c_m are the constants to be determined by matching this with the remaining boundary condition: $u(r, \theta, 0) = T_0$. Now,

$$u(r, \theta, 0) = T_0 = \sum_{m=1}^{\infty} c_m J_0(k_m r).$$

Multiplying $r J_0(k_\ell r)$ on both sides and integrate them in r, using the orthogonality condition, we have

$$T_0 \int_0^1 r J_0(k_\ell) \mathrm{d}r = c_\ell \int_0^1 r [J_0(k_\ell r)]^2 \mathrm{d}r.$$

Using the formulas provided in the front page, we have:

$$\frac{T_0}{k_\ell} J_1(k_\ell) = c_\ell \cdot \frac{1}{2} J_1^2(k_\ell), \quad \ell = 0, 1, \ldots,$$

Therefore, we have

$$c_m = \frac{2T_0}{k_m J_1(k_m)}.$$

Consequently, we have the final solution as follows:

$$\boxed{u(r, \theta, z) = 2T_0 \sum_{m=1}^{\infty} \frac{\mathrm{e}^{-k_m z}}{k_m J_1(k_m)} J_0(k_m r),}$$

where k_m is the zeros of J_0.

Problem 7 (20 pts) Consider a function $f(x) = x$ on the unit interval $[0, 1]$.

(a) (5 pts) Expand this in the Fourier series by viewing this as a periodic function with period 1.

Answer: Since the period is 1, if we expand x as

$$x \sim \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi nx) + b_n \sin(2\pi nx)),$$

then

$$c_n = \int_0^1 x e^{-i2\pi nx} dx = \frac{i}{2\pi n} \quad \text{by integration by parts for } n \neq 0\text{, and } c_0 = \frac{a_0}{2} = \int_0^1 x dx = \frac{1}{2}.$$

Converting a_n, b_n from c_n, we have:

$$a_n = c_n + c_{-n} = \frac{i}{2\pi n} + \frac{i}{-2\pi n} = 0, \quad n \neq 0,$$

$$b_n = i(c_n - c_{-n}) = -\frac{1}{\pi n}.$$

Therefore,

$$\boxed{x \sim \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{\pi n}.}$$

(b) (5 pts) Expand $f(x)$ in the Fourier cosine series by reflecting in an even manner at $x = 0$.

Answer: By even extension at $x = 0$, the function with reflection becomes an even function, and the period becomes 2 (i.e., the basic interval is now $[-1, 1]$). Thus, we can expand it as:

$$x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi nx),$$

where

$$\begin{aligned} a_n &= 2\int_0^1 x \cos(\pi nx) dx \\ &= 2\left[x \frac{\sin(\pi nx)}{\pi n}\right]_0^1 - 2\int_0^1 \frac{\sin(\pi nx)}{\pi n} dx \quad \text{via integration by parts.} \\ &= 2\frac{\cos(\pi n) - 1}{\pi^2 n^2} = 2\frac{(-1)^n - 1}{\pi^2 n^2}, \end{aligned}$$

for $n \neq 0$ and $a_0/2 = 1/2$ as part (a). Thus, we have:

$$\boxed{x \sim \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi^2 n^2} \cos(\pi nx) = \frac{1}{2} - 4\sum_{m=1}^{\infty} \frac{\cos(\pi(2m-1)x)}{\pi^2 (2m-1)^2}.}$$

(c) (5 pts) Expand $f(x)$ in the Fourier sine series by reflecting in an odd manner at $x = 0$.

Answer: By odd extension at $x = 0$, the function with reflection becomes an odd function, and the period becomes 2 (i.e., the basic interval is now $[-1, 1]$). Thus, we can expand it as:

$$x \sim \sum_{n=1}^{\infty} b_n \sin(\pi n x),$$

where

$$\begin{aligned} b_n &= 2\int_0^1 x \sin(\pi n x)\mathrm{d}x \\ &= 2\left[-x\frac{\cos(\pi n x)}{\pi n}\right]_0^1 + 2\int_0^1 \frac{\cos(\pi n x)}{\pi n}\mathrm{d}x \quad \text{via integration by parts.} \\ &= -2\frac{\cos(\pi n)}{\pi n} = 2\frac{(-1)^{n+1}}{\pi n}. \end{aligned}$$

Thus, we have:

$$\boxed{x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin(\pi n x).}$$

(d) (5 pts) Argue which of the above three expansions gives us the most faithful representation if we need to truncate the series in a finite number of terms.

Answer: If we compare the decay of the Fourier coefficients of the above three cases, we have the following:

$$\begin{aligned} a_n, b_n &\sim O(1/n) \quad \text{for part (a);} \\ a_n &\sim O(1/n^2) \quad \text{for part (b);} \\ b_n &\sim O(1/n) \quad \text{for part (c).} \end{aligned}$$

Therefore, clearly, part (b), i.e., the Fourier cosine series by reflecting a function in an even manner at $x = 0$, is the most faithful representation among these three.