Lecture 18: Basics of L^2 Theory II

Note Title

<u>Recap</u>: We already know that:

- $f \in PS(\mathbb{R})$, 2π-periodic $\Rightarrow S_N[f]$ conv. pointwise.
- $f \in PS(\mathbb{R}) \cap C(\mathbb{R})$, 2π-per $\Rightarrow S_N[f]$ conv. abs/unif.

<u>Question</u>: $f \in L^2[a,b] \Rightarrow \sum_1^\infty \langle f, \phi_n \rangle \phi_n \to f$ in norm?

<u>Lemma</u> If $f \in L^2[a,b]$, $\{\phi_n\}$: any ONset in $L^2[a,b]$,
<u>then</u> $\sum \langle f, \phi_n \rangle \phi_n$ conv. in norm and

$$\left\| \sum \langle f, \phi_n \rangle \phi_n \right\| \le \|f\|.$$

(Proof) By Bessel's ineq., $\forall f \in L^2[a,b]$,

$$\sum_1^\infty |\langle f, \phi_n \rangle|^2 \le \|f\|_2^2 < \infty.$$

So, by the Pythagorian Thm, we have

$$\left\| \sum_M^N \langle f, \phi_n \rangle \phi_n \right\|^2 = \sum_M^N \| \langle f, \phi_n \rangle \phi_n \|^2$$

$$= \sum_M^N |\langle f, \phi_n \rangle|^2 \underbrace{\|\phi_n\|^2}_{=1} \xrightarrow[M,N \to \infty]{} 0.$$

Thus, the partial sums of $\sum_1^\infty \langle f, \phi_n \rangle \phi_n$ form a Cauchy seq. in $L^2[a,b]$, which is complete.

$\Rightarrow \sum_1^\infty \langle f, \phi_n \rangle \phi_n$ conv. in norm to $\exists$ fcn $\in L^2[a,b]$

Finally, $\left\| \sum_1^\infty \langle f, \phi_n \rangle \phi_n \right\|^2 = \lim_{N \to \infty} \left\| \sum_1^N \langle f, \phi_n \rangle \phi_n \right\|^2$

$$= \lim_{N \to \infty} \sum_1^N |\langle f, \phi_n \rangle|^2$$

$$= \sum_1^\infty |\langle f, \phi_n \rangle|^2 \le \|f\|^2$$

Bessel's ineq.

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Remark: An example of an ON **set** in $L^2[-1,1]$ but not an ON**B** of $L^2[-1,1]$:

$$\phi_n(x) = \begin{cases} \sqrt{2}\sin n\pi x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } -1 \le x \le 0 . \end{cases}$$

Thm Let $\{\phi_n\}_1^\infty$ be an ON set in $L^2[a,b]$. Then the following cond's are equivalent:

(a) $\langle f, \phi_n \rangle = 0, \ \forall n \in \mathbb{N} \implies f \equiv 0$ (a.e.).

(b) $\forall f \in L^2[a,b], \ f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n$ in norm.

(c) $\forall f \in L^2[a,b], \ \|f\|^2 = \sum_1^\infty |\langle f, \phi_n \rangle|^2$ (Parseval's equality).

(Proof) (a) $\Rightarrow$ (b): By the lemma, $\sum_1^\infty \langle f, \phi_n \rangle \phi_n$ conv. in norm to a fcn in $L^2[a,b]$. Need to show that fcn is in fact f.

To do so, consider $g = f - \sum_1^\infty \langle f, \phi_n \rangle \phi_n$.

Take the inner prod. with ϕ_m:

$$\begin{aligned} \Rightarrow \langle g, \phi_m \rangle &= \langle f, \phi_m \rangle - \sum_1^\infty \langle f, \phi_n \rangle \underbrace{\langle \phi_n, \phi_m \rangle}_{= \delta_{n,m}} \\ &= \langle f, \phi_m \rangle - \langle f, \phi_m \rangle \\ &= 0, \ \forall m \in \mathbb{N}. \end{aligned}$$

By (a), $g \equiv 0$, i.e., $f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n$ (a.e.) //

(b) $\Rightarrow$ (c): $f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n$.

$$\begin{aligned} \|f\|^2 &= \lim_{N\to\infty} \Big\| \sum_1^N \langle f, \phi_n \rangle \phi_n \Big\|^2 = \lim_{N\to\infty} \sum_1^N \| \langle f, \phi_n \rangle \phi_n \|^2 \\ &= \sum_1^\infty |\langle f, \phi_n \rangle|^2 . \ // \end{aligned}$$

Pythagoras!

(c) $\Rightarrow$ (a): If $\|f\|^2 = \sum_1^\infty |\langle f, \phi_n \rangle|^2$ & $\langle f, \phi_n \rangle = 0, \forall n \in \mathbb{N}$, then $\|f\| = 0$ is a must. So, $f \equiv 0$ (a.e.). ///

Def. An ONset $\{\phi_n\}_1^\infty$ satisfying (a)–(c) of the thm is called a complete ON set (CONS) or ON Basis (ONB) of $L^2[a,b]$. $\{\langle f, \phi_n\rangle\}_1^\infty$ are called the (generalized) Fourier coef's of f w.r.t. $\{\phi_n\}_1^\infty$.
$\sum_1^\infty \langle f, \phi_n\rangle \phi_n$ is called the (generalized) Fourier series of f.

Note that an Orthogonal Basis (OB) can always be transformed to ONB by simple normalization.

Thm

$$
\begin{aligned}
&\{e^{inx}\}_{n\in\mathbb{Z}} : && \text{OB for } L^2[-\pi,\pi] \\
&\left\{\tfrac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n\in\mathbb{Z}} : && \text{ONB} \quad " \\
&\{\cos nx\}_{n=0}^\infty \cup \{\sin nx\}_{n=1}^\infty : && \text{OB} \quad " \\
&\left\{\tfrac{1}{\sqrt{2\pi}}\right\} \cup \left\{\tfrac{1}{\sqrt{\pi}}\cos nx\right\}_{n=1}^\infty \cup \left\{\tfrac{1}{\sqrt{\pi}}\sin nx\right\}_{n=1}^\infty : && \text{ONB} \quad " \\
&\{\cos nx\}_{n=0}^\infty : && \text{OB for } L^2[0,\pi] \\
&\left\{\tfrac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\tfrac{2}{\pi}}\cos nx\right\}_{n=1}^\infty : && \text{ONB} \quad " \\
&\{\sin nx\}_{n=1}^\infty : && \text{OB} \quad " \\
&\left\{\sqrt{\tfrac{2}{\pi}}\sin nx\right\}_{n=1}^\infty : && \text{ONB} \quad "
\end{aligned}
$$

(Proof) We'll prove this for $\phi_n = \frac{1}{\sqrt{2\pi}} e^{inx}$.
The other cases are similar.
We already know these ϕ_n's form an ONset.
The only thing we need to show: its completeness!
i.e., To Show: $\forall f \in L^2[-\pi,\pi]$, $\|f - \sum_{-N}^{N} \langle f, \phi_n\rangle \phi_n\| \underset{N\to\infty}{\longrightarrow} 0$
(If so, we know $\{\phi_n\}$: an ONB thanks to the Thm.)
Now, $\forall f \in L^2[-\pi,\pi]$, $\forall \varepsilon > 0$, $\exists \tilde{f} \in C^\infty[-\pi,\pi]$ s.t.
$\|f - \tilde{f}\| \le \varepsilon$ via the Thm in Lec. 17.

So, $\| f - \sum_{-N}^{N} \langle f, \phi_n \rangle \phi_n \|$

$$= \| f - \tilde{f} + \tilde{f} - \sum_{-N}^{N} \langle \tilde{f}, \phi_n \rangle \phi_n + \sum_{-N}^{N} \langle \tilde{f}, \phi_n \rangle \phi_n - \sum_{-N}^{N} \langle f, \phi_n \rangle \phi_n \|$$

$$\leq \| f - \tilde{f} \| + \| \tilde{f} - \sum_{-N}^{N} \langle \tilde{f}, \phi_n \rangle \phi_n \| + \| \sum_{-N}^{N} \langle \tilde{f} - f, \phi_n \rangle \phi_n \| \quad \text{Pythagoras}$$

$$\leq \varepsilon + \leq \varepsilon + \left(\sum_{-N}^{N} | \langle \tilde{f} - f, \phi_n \rangle |^2 \right)^{\frac{1}{2}}$$

for $N \geq {}^{\exists} N_0(\varepsilon)$ because $\sum_{-N}^{N} \langle \tilde{f}, \phi_n \rangle \phi_n \xrightarrow[\text{unif.}]{} \tilde{f}$

$$\leq 2\varepsilon + \| \tilde{f} - f \| \leq 3\varepsilon. \quad ///$$

Bessel's ineq.

☆ Another Example of ONB on $L^2[-1,1]$:

Legendre Polynomials

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n = 0, 1, 2, \cdots$$

called the Rodrigues formula.

is called the n th Legendre polynomial.

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $\cdots$

Thm The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ form an OB for $L^2[-1,1]$, with $\|P_n\|^2 = \frac{2}{2n+1}$.

So, $\left\{ \sqrt{n + \frac{1}{2}}\, P_n(x) \right\}_{n=0}^{\infty}$ form an ONB for $L^2[-1,1]$.

These can also be obtained by applying the Gram-Schmidt procedure to $\{1, x, x^2, x^3, \cdots\}$.

Let $\varphi_n(x) := \sqrt{n + \frac{1}{2}}\, P_n(x)$

For a given $f \in L^2[-1,1]$, suppose we have

$$f = \sum_0^\infty \alpha_n \varphi_n = \sum_0^\infty c_n P_n$$

Then, the relationship between α_n & c_n is

$$\sum \alpha_n \varphi_n = \sum \underbrace{\alpha_n \sqrt{n+\tfrac{1}{2}}}_{= c_n} P_n$$

and $\alpha_n = \langle f, \varphi_n \rangle$

$$\Rightarrow c_n = \sqrt{n+\tfrac{1}{2}}\,\alpha_n = \sqrt{n+\tfrac{1}{2}}\langle f, \varphi_n \rangle$$
$$= \sqrt{n+\tfrac{1}{2}}\langle f, \sqrt{n+\tfrac{1}{2}} P_n \rangle$$
$$= (n+\tfrac{1}{2})\langle f, P_n \rangle.$$

Of course, dealing with an ONB $\{\varphi_n\}$ is easier and more convenient.

Example: Expand $\chi_{[0,1]}(x)$ into a Legendre series!

$$\chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$c_n = (n+\tfrac{1}{2})\int_{-1}^{1} \chi_{[0,1]}(x) P_n(x)\,dx$$
$$= (n+\tfrac{1}{2})\int_0^1 P_n(x)\,dx$$
$$= \frac{2n+1}{2^{n+1} n!}\int_0^1 \frac{d^n}{dx^n}(x^2-1)^n\,dx$$
$$= \frac{2n+1}{2^{n+1} n!}\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_0^1$$
$$= -\frac{2n+1}{2^{n+1} n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\Big|_{x=0}$$
$$\underset{\substack{\uparrow \\ \text{the Binomial Thm}}}{=} -\frac{2n+1}{2^{n+1} n!}\frac{d^{n-1}}{dx^{n-1}}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^{2k}\right)\Big|_{x=0}$$

Since $\frac{d^{n-1}}{dx^{n-1}} x^{2k} = 2k(2k-1)\cdots(2k-n+2)x^{2k-n+1}$ and we evaluate it at $x=0$, $C_n = 0$ unless $2k = n-1$ or $n=0$.

$\Leftrightarrow n = 2k+1$

$$C_0 = \frac{1}{2}\int_0^1 \underbrace{P_0(x)}_{=1}\,dx = \frac{1}{2}.$$

$$C_{2k+1} = -\frac{4k+3}{2^{2k+2}(2k+1)!}(-1)^{2k+1-k}\binom{2k+1}{k}(2k)!$$

$$= (-1)^k \frac{4k+3}{4^{k+1}} \frac{(2k)!}{k!\,(k+1)!}$$

$$\Rightarrow \chi_{[0,1]}(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k(4k+3)(2k)!}{4^{k+1}\,k!\,(k+1)!} \underbrace{P_{2k+1}(x)}_{\text{odd only}} \quad //$$

* Interesting fact

Given a fcn $f \in L^2[-1,1]$, the best n-deg. poly. over $[-1,1]$ is given by $\sum_{k=0}^{n} c_k P_k(x)$, $c_k = (k+\frac{1}{2})\langle f, P_k\rangle$.

in the least squares sense.

$$\min_{p\in P_n} \|f-p\|^2 = \|f - \sum_{k=0}^{n} c_k P_k\|^2$$

Why? $\|f-p\|^2 = \|\sum_0^\infty c_k P_k - p\|^2 = \|\sum_0^\infty c_k P_k - \sum_0^n b_k P_k\|^2$

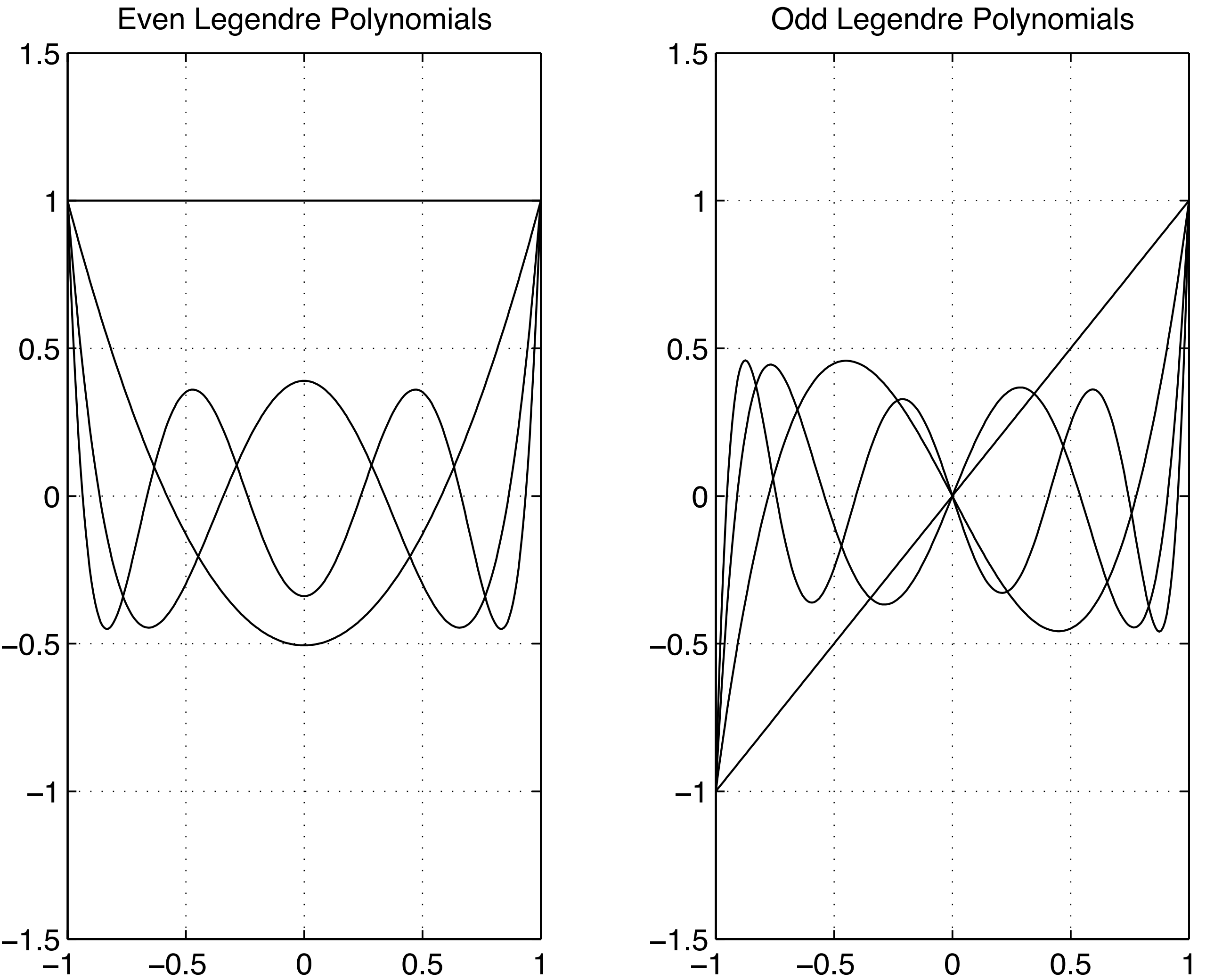
$$= \|\sum_0^n (c_k - b_k)P_k + \sum_{n+1}^\infty c_k P_k\|^2$$

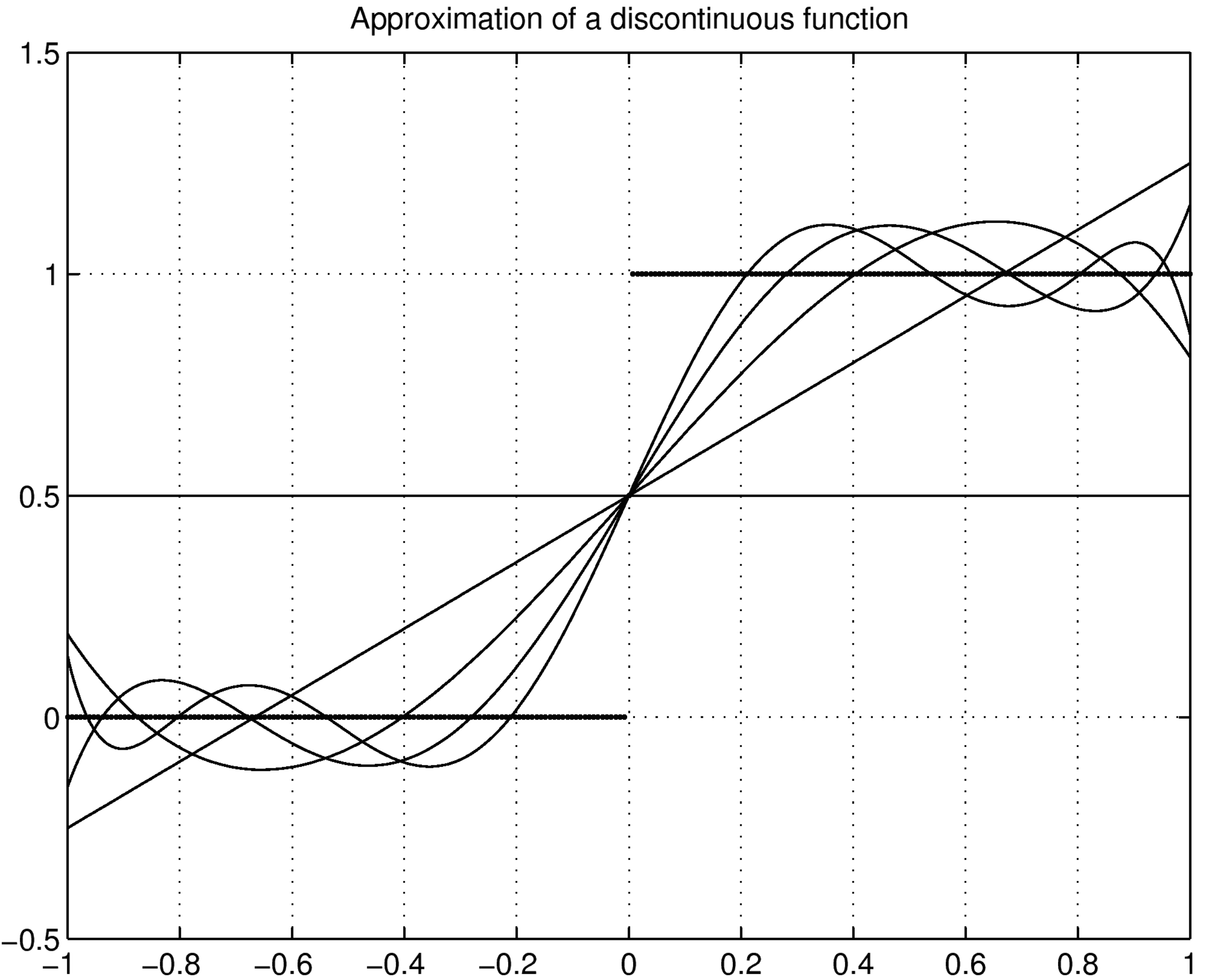
Pythagoras

$$= \|\sum_0^n (c_k-b_k)P_k\|^2 + \|\sum_{n+1}^\infty c_k P_k\|^2$$

$$\geq \|\sum_{n+1}^\infty c_k P_k\|^2$$

$=$ iff $b_k = c_k$, $k = 0, 1, \cdots, n$. //


Even Legendre Polynomials
Odd Legendre Polynomials
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1



Approximation of a discontinuous function
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1