Lecture 22: Green's Functions I

Note Title

- Want to diagonalize the self-adj. op. $\mathcal{L}$ arised in the Sturm–Liouville theory. ↳ Differential op.

- If $\mathcal{L}$ were a compact self-adj. op., then it's pretty close to a matrix, i.e., one can apply tools similar to those in linear algebra (e.g., eigenvalue decomposition, etc.)

- Unfortunately, $\mathcal{L}$ is far from being compact! In fact, $\mathcal{L}$ is unbounded!

  Ex. Let $\mathcal{L} = \frac{d^2}{dx^2}$ on $[0,1]$ with $D(\mathcal{L}) = \{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \}$.

  Then, $\|\mathcal{L}\| := \sup\limits_{f \in D(\mathcal{L})} \frac{\|\mathcal{L} f\|_2}{\|f\|_2}$

  Take $f_n = \sqrt{2} \sin n\pi x \in D(\mathcal{L})$. $\|f_n\|_2 = 1$.

  $\Rightarrow \mathcal{L} f_n = -(n\pi)^2 \sqrt{2} \sin n\pi x$. $\|\mathcal{L} f_n\|_2 = (n\pi)^2$.

  $\Rightarrow \|\mathcal{L} f_n\| / \|f_n\| = (n\pi)^2 \longrightarrow \infty$!

  Hence $\|\mathcal{L}\| = \infty$, i.e., unbdd. //

- Therefore, we investigate $\mathcal{L}^{-1}$ instead. $\mathcal{L}^{-1}$ is an integral op., and turns out to be compact in our case!

- We'll review some basic properties of compact op's later. But first, let's proceed with deriving $\mathcal{L}^{-1}$!

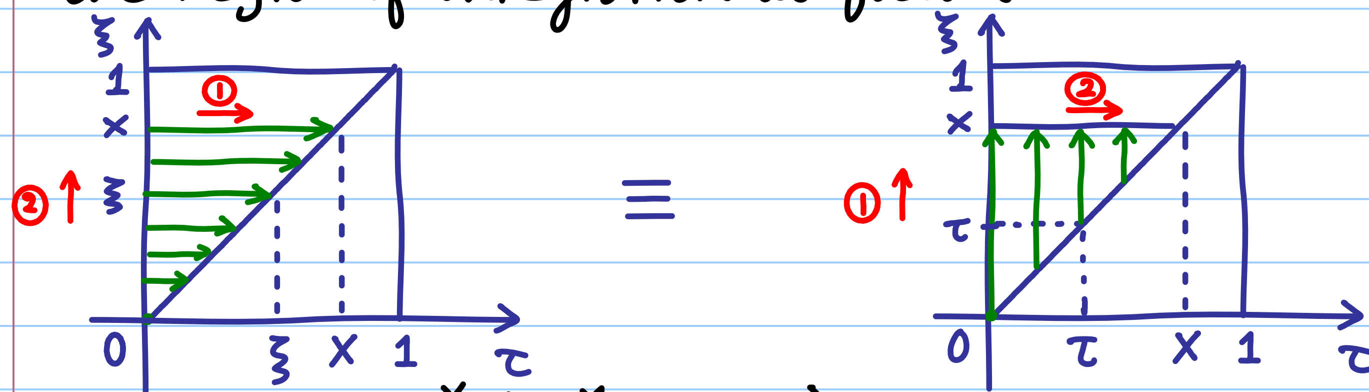
Consider an RSL: $\mathcal{L} f = g$, i.e., $(pf')' + qf = g$, where $D(\mathcal{L}) = \{ f \in H^2(a,b) \mid \alpha f(a) + \alpha' f'(a) = 0, \beta f(b) + \beta' f'(b) = 0 \}$, $g \in L^2(a,b)$. Want to do: $f = \mathcal{L}^{-1} g$.

Let's look at the simplest case first, i.e., $\mathcal{L} = \frac{d^2}{dx^2}$ on $[0,1]$ with the Dirichlet B.C.: $f(0) = f(1) = 0$.

Integrate $\mathcal{L}f = f'' = g$ twice to get

$$f(x) = \int_0^x \left(\int_0^\xi g(\tau)\,d\tau \right) d\xi + \underline{A}x + \underline{B}.$$ integration const's

Let's swap the order of integration by considering the region of integration as follows:



So, $$f(x) = \int_0^x \left(\int_\tau^x g(\tau)\,d\xi \right) d\tau + Ax + B$$ → a fcn of τ!

$$= \int_0^x (x-\tau)\,g(\tau)\,d\tau + Ax + B$$

Now, $f(0) = B = 0$

$$f(1) = \int_0^1 (1-\tau)\,g(\tau)\,d\tau + A = 0.$$

$$\Rightarrow f(x) = \int_0^x (x-\tau)\,g(\tau)\,d\tau - x\int_0^1 (1-\tau)\,g(\tau)\,d\tau$$

$$= \int_0^x (x-\tau)\,g(\tau)\,d\tau - \left\{ \int_0^x x(1-\tau)\,g(\tau)\,d\tau + \int_x^1 x(1-\tau)\,g(\tau)\,d\tau \right\}$$

$$= \int_0^x \tau(x-1)\,g(\tau)\,d\tau + \int_x^1 x(\tau-1)\,g(\tau)\,d\tau$$

$$=: \int_0^1 k(x,\tau)\,g(\tau)\,d\tau =: \mathcal{K}g(x)$$

where $$k(x,\tau) := \begin{cases} \tau(x-1) & \text{if } 0 \le \tau \le x; \\ x(\tau-1) & \text{if } x \le \tau \le 1. \end{cases}$$

$\mathcal{K}$: an integral op. ⇒ Mathematica fig. show here.

$k(\cdot,\cdot)$: the kernel fcn of $\mathcal{K}$.

In this case, $k(\cdot,\cdot)$ has a special name:
Green's fcn for $\mathcal{L}$ (or the BVP associated w. $\mathcal{L}$)

Now, for a general case of RSL: $\mathcal{L}f = g$, $f \in D(\mathcal{L})$, we use the method of variation of parameters due to Lagrange. First, let's define an extended op.:

$$\mathcal{M} := \frac{d}{dx}\left(p\frac{d}{dx}\cdot\right) + q\,\cdot\ , \quad D(\mathcal{M}) = \underbrace{H^2(a,b)}_{\text{no explicit B.C.}} \supset D(\mathcal{L})$$

Now suppose we can find two linearly indep. sol's u, v of the corresponding homogeneous eqn. $\mathcal{M}f = 0$. Then we look for a sol. of $\mathcal{L}f = g$ of the form:

$$f = \varphi u + \psi v, \quad {}^{\exists}\varphi, \psi \in C^1[a,b].$$

$$\Rightarrow f' = \varphi u' + \psi v' + \varphi' u + \psi' v$$

Let's choose φ, ψ s.t. $\varphi' u + \psi' v = 0$ ——— ①

$$\Rightarrow f' = \varphi u' + \psi v'$$

$$\begin{aligned}
\Rightarrow \mathcal{L}f &= (pf')' + qf \\
&= p'\varphi u' + p'\psi v' + p\varphi' u' + p\varphi u'' + p\psi' v' + p\psi v'' \\
&= \varphi' p u' + \varphi\underbrace{(p'u' + pu'' + qu)}_{= \mathcal{M}u = 0} + q\varphi u + q\psi v \\
&\quad + \psi' p v' + \psi\underbrace{(p'v' + pv'' + qv)}_{= \mathcal{M}v = 0} \\
&= p(\varphi' u' + \psi' v') \text{ ——— ②}
\end{aligned}$$

① $\times pu'$ − ② $\times u$: $\quad p(\cancel{\varphi' u' u} + \psi' u v) - p(\cancel{\varphi' u' u} + \psi' u v') = -ug$

i.e., $\psi' p(uv' - vu') = ug$ ——— ③

By Lagrange's identity, we have

$$u\underbrace{\mathcal{M}v}_{=0} - v\underbrace{\mathcal{M}u}_{=0} = (p(uv' - vu'))' = 0.$$

So, $p(uv' - vu') = $ const., say, c.

$\Rightarrow$ ③ becomes $c\psi' = ug$.

Similarly, we can also get $c\varphi' = -vg$.

So, assuming $c \neq 0$, we can set

$$\varphi(x) = \frac{1}{c}\left(\int_x^b v(\tau)g(\tau)\,d\tau + \underline{A}\right)$$

$$\psi(x) = \frac{1}{c}\left(\int_a^x u(\tau)g(\tau)\,d\tau + \underline{B}\right)$$

→ Integration const's

as long as $g \in C[a,b]$, $\varphi, \psi \in C^1[a,b]$ and satisfy $c\varphi' = -vg$, $c\psi' = ug$.
And in turn, $\varphi' u + \psi' v = 0$ and $f = \varphi u + \psi v$ is a sol. of $\mathcal{L}f = g$. Using the B.C., we expect A, B to be determined.
Yet, $\exists$ two potential problems:
ⓐ Can we really assume $c \neq 0$?
ⓑ Can we extend the result to $g \in L^2[a,b]$ from $g \in C[a,b]$?
We first deal with ⓐ and briefly discuss ⓑ later. (full discussion on ⓑ requires the knowledge of measure theory.)
Let's proceed! $Mf = 0$ has nontrivial (i.e., nonzero) sol's $f = u$, s.t. $\alpha u(a) + \alpha' u'(a) = 0$
$f = v$, s.t. $\beta v(b) + \beta' v'(b) = 0$
thanks to the Existence Thm for 2nd order ODEs.
Now set $f(x) = \varphi(x)u(x) + \psi(x)v(x)$, $f' = \varphi u' + \psi v'$

$$= \frac{1}{c}u(x)\left\{\int_x^b v(\tau)g(\tau)\,d\tau + A\right\} + \frac{1}{c}v(x)\left\{\int_a^x u(\tau)g(\tau)\,d\tau + B\right\}$$

By our derivation, we know $\mathcal{L}f = g$, but f must satisfy the B.C. at both ends.

$$f(a) = \frac{1}{c}u(a)\left\{\int_a^b v(\tau)g(\tau)\,d\tau + A\right\} + \frac{1}{c}v(a)B$$

$$f'(a) = \varphi(a)u'(a) + \underbrace{\psi(a)}_{=B/c}v'(a) = \frac{1}{c}u'(a)\left\{\int_a^b v(\tau)g(\tau)\,d\tau + A\right\} + \frac{1}{c}v'(a)B$$

So, $0 = \alpha f(a) + \alpha' f'(a) = \frac{1}{c}\left\{\int_a^b v(\tau)g(\tau)d\tau + A\right\}\underbrace{(\alpha u(a) + \alpha' u'(a))}_{=0}$
$+ \frac{B}{c}\underbrace{(\alpha v(a) + \alpha' v'(a))}_{\neq 0}$

$\Rightarrow B = 0$. Similarly, we get $A = 0$.

So, we now have:
If $f = \varphi u + \psi v$ with $\begin{cases} \varphi(x) = \frac{1}{c}\int_x^b v(\tau)g(\tau)d\tau \\ \psi(x) = \frac{1}{c}\int_a^x u(\tau)g(\tau)d\tau \end{cases}$ $x \in [a,b]$,

then f is a sol. of $\mathcal{L}f = g$ satisfying the B.C.'s!

Now, we'll prove the following thm, which takes care of (a):

Thm Suppose that 0 is not an eigenval. of the RSL. Then $\forall g \in C[a,b]$, $\mathcal{L}f = g$ (incl. B.C.'s) has the unique sol. $f(x) = \int_a^b k(x,\tau)g(\tau)d\tau$

where $k(x,\tau) := \begin{cases} \frac{1}{c}v(x)u(\tau) & \text{if } a \le \tau \le x \le b \\ \frac{1}{c}u(x)v(\tau) & \text{if } a \le x \le \tau \le b, \end{cases}$

u, v are non-zero real sol's of $\underbrace{\mathcal{M}_1}_{=\mathcal{M}+\text{B.C. at } a} u = 0$, $\underbrace{\mathcal{M}_2}_{=\mathcal{M}+\text{B.C. at } b} v = 0$,

and $c = p(uv' - vu') = \text{const}$.

Futhermore, f'' exists and $f'' \in C[a,b]$. //

First, we prove:

Lemma Under the assumption of the above thm,
$uv' - vu' \neq 0 \quad \forall x \in [a,b]$. $=: W[u,v](x)$ Wronskian!

(Proof) Suppose $uv' - vu' = \underbrace{\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}}$ $= 0$ at $\exists x_0 \in [a,b]$.

Then $\exists (\gamma, \delta) \neq (0,0)$ s.t.

$\begin{cases} \gamma u(x_0) + \delta v(x_0) = 0 \\ \gamma u'(x_0) + \delta v'(x_0) = 0 \end{cases} \leftarrow \begin{bmatrix} u(x_0) & v(x_0) \\ u'(x_0) & v'(x_0) \end{bmatrix}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow f = \gamma u + \delta v$ is a sol. of $\mathcal{M}f = 0$ with $\begin{cases} f(x_0) = 0 \\ f'(x_0) = 0. \end{cases}$

$\Rightarrow$ By the Fundamental Existence Thm for 2nd order ODEs, $f(x) \equiv 0 \quad \forall x \in [a,b]$.

$\Rightarrow \gamma u + \delta v \equiv 0$, but $(\gamma, \delta) \not\equiv (0,0)$

$\Leftrightarrow u, v$: linearly dep. and $u = -\frac{\delta}{\gamma} v$.

$\Rightarrow \beta u(b) + \beta' u'(b) = 0$ since $\beta v(b) + \beta' v'(b) = 0$.

$\Rightarrow u$ satisfies $\mathcal{L} u = 0$, $u \not\equiv 0$.

$\Leftrightarrow u$: an eigenfcn of $\mathcal{L}$ corresp. to 0 eigenvalue, which contradicts with the assumption! #

(Proof of Thm) $p > 0$ & $uv' - vu' \neq 0$, $\forall x \in [a,b]$ via Lemma.

$\Rightarrow c = p(uv' - vu') \neq 0$.

Since 0 is not an eigval of $\mathcal{L}$, $\mathcal{L} f = g$ certainly has at most one sol. in $D(\mathcal{L})$. The foregoing calculations have already shown that $f = \varphi u + \psi v$ is such a sol.

It remains to check that $f \in C^2[a,b]$.

u'' exists since u is a sol. to $M_1 f = 0$ (obvious!)

and $u'' = -\frac{p' u'}{p} - \frac{q u}{p} \in C[a,b] \Rightarrow u \in C^2[a,b]$.

Similarly, $v \in C^2[a,b]$.

Now, $\varphi', \psi' \in C[a,b]$ because $c\psi' = ug$, $c\varphi' = -vg$.

Since $f' = \varphi u' + \psi v' \in C^1[a,b]$, i.e., $f'' \in C[a,b]$. ///