Lecture 25: Eigenfunction Expansion

Note Title

☆ Sturm-Liouville Thm

• Separated B.C.

• $0 \in \sigma(\mathcal{L})$ could happen.

The RSL system has an infinite seq. $\{\lambda_j\}_{j\in\mathbb{N}}$ of eigval's. Each eigval is real and simple, and $|\lambda_j| \to \infty$ as $j \to \infty$.
If φ_j is an eig fcn of the RSL system corresp. to λ_j then $\{\sqrt{w}\,\varphi_j\}_{j\in\mathbb{N}}$ is an ONB of $L^2(a,b)$.
i.e., $\{\varphi_j\}_{j\in\mathbb{N}}$ is an ONB of $L^2_w(a,b)$.

(Sketch of Proof)

① First, assume that $0 \notin \sigma(\mathcal{L})$

Step 1.1: $w(x) \equiv 1$ on $[a,b]$. Since $\underbrace{(pf')' + qf}_{\mathcal{L}f} = -\lambda f$, we have $\lambda \in \sigma(\text{RSL}) \Leftrightarrow -\lambda \in \sigma(\mathcal{L})$.

Now K: a compact self-adj. op. and shares the eigfcns with $\mathcal{L}$. $\lambda \in \sigma(\mathcal{L})$, $1/\lambda \in \sigma(K)$.

$\Rightarrow$ By the Spectral Thm, $\exists$ ON seq. $\{\varphi_j\}$ of K. Let $\{\mu_j\}$ be the corresp. eigval's.

$\Rightarrow$ By the $K \leftrightarrow \mathcal{L}$ Thm, $\{1/\mu_j\} \subset \sigma(\mathcal{L})$.

$\Rightarrow$ The eigval's of RSL system are $\lambda_j = -1/\mu_j \in \mathbb{R}$ and $\mu_j \to 0$ due to the compactness of K.

$\Rightarrow |\lambda_j| \to \infty$, and the corresp. eigfcn's $\{\varphi_j\}$ form an ONB of $L^2(a,b)$, which is separable.

Step 1.2: $w(x) > 0$ and $\in C[a,b]$.

Informally speaking, let $\lambda \in \sigma(\text{RSL})$ with φ as the eigfcn.

Then, $\mathcal{L}\varphi = -\lambda w \varphi \iff -\lambda^{-1}\varphi = \mathcal{L}^{-1} w \varphi$ $\quad \lambda \neq 0$

$$\iff -\lambda^{-1}\sqrt{w}\,\varphi = \sqrt{w}\,\mathcal{L}^{-1} w\,\varphi$$

$$\iff -\lambda^{-1}\sqrt{w}\,\varphi = (\sqrt{w}\,\mathcal{L}^{-1}\sqrt{w})\sqrt{w}\,\varphi$$

$$\iff -\lambda^{-1} \in \sigma(\sqrt{w}\,\mathcal{L}^{-1}\sqrt{w})$$

with $\sqrt{w}\,\varphi$ as the eigfcn.

Step 1.3 : Simplicity of each eigval for the RSL sys (with the separated B.C.'s) was already proved before.

② The case when $0 \in \sigma(\mathcal{L})$: Pick $\mu \in \mathbb{R}$ s.t. $\mu \notin \sigma(\mathrm{RSL})$. This is possible since $\sigma(\mathrm{RSL})$ is countable. Let RSL_μ be an RSL by replacing q by $q + \mu w$.

Then, (f, λ) is an eigenpair of RSL_μ

$$\Longleftrightarrow (pf')' + (q + \mu w + \lambda w) f = 0$$

This is possible iff $\lambda + \mu \in \sigma(\mathrm{RSL})$

Hence $0 \notin \sigma(\mathrm{RSL}_\mu)$, i.e., we can always shift the eigval's. So ① can be applied here. ///

Remark : The above thm is just a starting point of a major branch of analysis. For any physical systems that can be described/modeled by the RSL systems, it is important to know the distribution of their eigval's and the behavior of the corresp. eigfcn's. The following are samples of the facts on the general RSL:

- $\lambda_j \to \infty$ as $j \to \infty$ (i.e., only finitely many λ_j's are < 0);
- $\sum_{\lambda_j \neq 0} \frac{1}{\lambda_j}$ converges ;
- φ_j has exactly j zeros on $[a, b]$.

See, e.g., Titchmarsh (1962) and/or Amrein et al. (2005).

We could not discuss the SSL systems in details. Consult the above books as well as Yosida (1991) and Stakgold & Holst (2011, Chap. 7).

☆ Solution of the Hanging Chain (HC) Problem

What use is it to know that a given SL system has an ONB of $L^2(a,b)$ consisting of its eigfcn's?
$\Rightarrow$ Can write down the solution of the IV-BV Problem from which the SL system arose. That is, we can justify the separation of variables!

To illustrate this, let's recall the HC problem of Lecture 20.

$$\text{(HC)}\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) \\ \text{B.C.}: \ u(\ell,t)=0, \quad 0\le t<\infty. \\ \text{I.C.}: \ u(x,0)=u_0(x), \ \dfrac{\partial u}{\partial t}(x,0)=0, \quad 0\le x\le \ell. \\ \text{BDD.}: \ \sup|u(x,t)|<\infty, \quad 0\le x\le\ell, \ 0\le t<\infty. \end{cases}$$

The corresp. SSL system is:

$$\text{(HCSL)}\begin{cases} (xf')' + \lambda f = 0, \quad 0\le x\le \ell. \\ \text{B.C.}: \ f(\ell)=0. \\ \text{BDD.}: \ f: \text{bdd on } (0,\ell]. \end{cases}$$

<u>Lemma</u> All the eigval's of HCSL are positive.

(Proof) Let (λ,φ) be an eigenpair of HCSL with $\|\varphi\|_2 = 1$. Let $\mathcal{L}f := (xf')'$, i.e., $(-\lambda,\varphi)$ be an eigenpair of $\mathcal{L}$ if (λ,φ) is an eigenpair of HCSL.

Now, $\lambda = -\langle \mathcal{L}\varphi,\varphi\rangle = -\int_0^\ell (x\varphi')'\,\overline{\varphi}\,dx$

$$= -\underbrace{\cancel{[x\varphi'\overline{\varphi}]_0^\ell}}_{0} + \int_0^\ell x|\varphi'|^2\,dx > 0. \quad ///$$

Let (λ_j,φ_j), $j\in\mathbb{N}$, be the eigenpairs of HCSL with $\|\varphi_j\|_2 = 1$, $\forall j\in\mathbb{N}$.

As we discussed in Lecture 20, we hope to express u as an infinite sum of normal modes:

(*) $u(x,t) = \sum_1^\infty c_j \varphi_j(x) \cos\sqrt{\lambda_j}\, t$

If this is true $\forall t \geq 0$, then

$u(x,0) = u_0(x) = \sum_1^\infty c_j \varphi_j(x) \leftarrow$ $\{c_j\}$: the Fourier coef's of u_0 w.r.t. $\{\varphi_j\}$.

$\Rightarrow$ This holds if $\{\varphi_j\}$ forms an ONB of $L^2(0,\ell)$.

Unfortunately, HCSL is singular, so the thm's for the RSL cannot be applied. Much more work is needed to show that $\{\varphi_j\}$: complete in $L^2(0,\ell)$; see Watson (1944).

Let's proceed by assuming that $\{\varphi_j\}$: an ONB for $L^2(0,\ell)$. Then we expect the sol. of HC is (*).

If term-by-term differentiation of (*) is valid, then

$$\frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) = \sum_1^\infty c_j \frac{d}{dx}\left(x \frac{d\varphi_j}{dx}\right) \cdot \cos\sqrt{\lambda_j}\, t = -\sum_1^\infty c_j \lambda_j \varphi_j(x) \cos\sqrt{\lambda_j}\, t.$$

Similarly $\frac{\partial^2 u}{\partial t^2}$ leads to the same series as above.

However, does the RHS make sense with $\lambda_j \to \infty$?

Nevertheless, (*) is justified via:

Thm Let $u_0 \in C^2[0,\ell]$. Then, $\exists$ at most one sol. of HC $u(x,t) \in C^2([0,\ell]\times[0,\infty))$. If $\exists$ a sol., then it is given by

(**) $u(\cdot,t) = \sum_1^\infty c_j \cdot \cos\sqrt{\lambda_j}\, t \cdot \varphi_j$ viewed as a fcn on $[0,\ell]$,

where (λ_j, φ_j), $j \in \mathbb{N}$ are the eigenpairs of HCSL with $\|\varphi_j\|_2 = 1$ and $c_j = \langle u_0, \varphi_j \rangle$, $\forall j \in \mathbb{N}$. (**) holds w.r.t. $\|\cdot\|_2 = \|\cdot\|_{L^2(0,\ell)}$ for every $t \geq 0$.

(Proof) Let u be a sol. of HC. For each $t \geq 0$, let $c_j(t) := \langle u(\cdot,t), \varphi_j \rangle$. Since $\{\varphi_j\}_{j\in\mathbb{N}}$ form an ONB of $L^2(0,\ell)$, we have

(***) $u(\cdot,t) = \sum_1^\infty c_j(t)\varphi_j$ in $L^2(0,\ell)$.

<u>Claim</u>: $c_j(\cdot) \in C^2[0,\infty)$ and $\ddot{c}_j(t) = \langle u_{tt}(\cdot,t), \varphi_j \rangle$.
We'll prove this claim later. Suppose this claim holds.
Then since u is a sol. of HC,

$$\ddot{c}_j(t) = \left\langle \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)(\cdot,t), \varphi_j \right\rangle = \langle \mathcal{L}u(\cdot,t), \varphi_j \rangle$$

$$\overset{\mathcal{L}:\text{ self-adj.}}{=} \langle u(\cdot,t), \mathcal{L}\varphi_j \rangle$$

$$= \langle u(\cdot,t), -\lambda_j\varphi_j \rangle$$

$$= -\lambda_j c_j(t).$$

So, $c_j(\cdot)$ satisfies the eqn. of simple harmonic motion whose general sol. is of the form:
$c_j(t) = A_j \cos\sqrt{\lambda_j}\, t + B_j \sin\sqrt{\lambda_j}\, t$, A_j, B_j: arb. const's.
Now we use the I.C.!
Since $\dot{c}_j(t) = \langle u_t(\cdot,t), \varphi_j \rangle$,

$$\dot{c}_j(0) = \langle u_t(\cdot,0), \varphi_j \rangle = \langle 0, \varphi_j \rangle = 0.$$

$\Rightarrow B_j = 0$, $\forall j \in \mathbb{N}$.
Now, $A_j = c_j(0) = \langle u(\cdot,0), \varphi_j \rangle = \langle u_0, \varphi_j \rangle = c_j$
So, $c_j(t) = c_j \cos\sqrt{\lambda_j}\, t$, $j \in \mathbb{N}$.
Hence, by (***), we have

$$u(\cdot,t) = \sum_1^\infty c_j \cdot \cos\sqrt{\lambda_j}\, t \cdot \varphi_j \text{ in } L^2(0,\ell). \quad ///$$

<u>Remark</u>: The C^2 condition on $u_0(\cdot)$ and $u(\cdot,\cdot)$ may be too restrictive, especially considering realistic problems. What we have discussed are the so-called classical solutions. For less restrictive conditions, we need the notion of weak solutions and the theory of Sobolev spaces. See, e.g., Folland (1995), Lieb & Loss (2001), ...

(Proof of the Claim) For $t \geq 0$ & $h \geq -t$, we have

$$\frac{c_j(t+h)-c_j(t)}{h} - \langle u_t(\cdot,t), \varphi_j \rangle$$

$$= \left\langle \frac{u(\cdot,t+h)-u(\cdot,t)}{h} - u_t(\cdot,t), \varphi_j \right\rangle$$

$$= \int_0^\ell \underbrace{\left\{ \frac{u(x,t+h)-u(x,t)}{h} - u_t(x,t) \right\}}_{\to 0 \text{ as } h \to 0 \text{ at fixed } t,\ \forall x \in [0,\ell].} \overline{\varphi_j(x)}\, dx. \quad \text{——} (\star)$$

So, *if* we can use the Dom. Conv. Thm., *then* the above leads to $\dot{c}_j(t) = \langle u_t(\cdot,t), \varphi_j \rangle$, and repeating the same argument, we can show $\ddot{c}_j(t) = \langle u_{tt}(\cdot,t), \varphi_j \rangle$, and $c_j(\cdot) \in C^2[0,\infty)$.

But in fact, the Dom. Conv. Thm. holds for $(\star)$!

$\because$ With t: fixed, let $M = M(t) := \sup\limits_{\substack{0 \leq t' \leq t+1 \\ 0 \leq x \leq \ell}} |u_t(x,t')|$.

Since $u_t(\cdot,\cdot) \in C([0,\ell]\times[0,\infty))$ via assump. and $[0,\ell]\times[0,t+1]$ is compact, $M < \infty$. Now, let $g(x) := 2M|\varphi_j(x)|$.

By the Mean Value Thm, $\left|\frac{u(x,t+h)-u(x)}{h}\right| = |u_t(x,t+\theta h)| \leq M$, $\exists \theta \in (0,1)$

provided that $-t \leq h \leq 1$.

$\Rightarrow$ For these values of h, the modulus of the integrand in $(\star) \leq g(x)$, and $\|g\|_2 < \infty$ since $g \in C[0,\ell]$ and bdd on $[0,\ell]$.

So, the Dom. Conv. Thm. does apply in $(\star)$. ///