An Example of the Method of Lagrange Multiplier

Due to the time shortage, I could not discuss the computational details of the following problem. Thus, I would like to provide you with the complete solution here.

Question: A rectangular box *without a lid* is to be made from 12m^2 of card board. Find the maximum volume of such a box.

Answer: Let x, y, z be the length, width, and height of such a box. The objective function that we want to maximize is the volume of this box:

$$f(x, y, z) = xyz.$$

The constraint we have is:

$$g(x, y, z) = xy + 2yz + 2zx - 12 = 0,$$

because we have to make the two pairs of side walls and one base, but we do not need the lid (top surface), which is why xy is not $2xy$ in the above equation.

Step 1: Compute all the points (x, y, z) that satisfy $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and the constraint $g(x, y, z) = 0$. The gradient vectors can be computed easily:

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle = \langle yz, zx, xy \rangle.$$

$$\nabla g(x, y, z) = \langle g_x, g_y, g_z \rangle = \langle y + 2z, x + 2z, 2y + 2x \rangle.$$

Thus, we have the following four equations:

$$\begin{aligned} yz &= \lambda(y + 2z), && (1) \\ zx &= \lambda(x + 2z), && (2) \\ xy &= \lambda(2y + 2x), && (3) \\ xy &+ 2yz + 2zx - 12 = 0. && (4) \end{aligned}$$

There are four unknowns and four equations. Hence we should be able to solve these. Multiplying x to the both sides of (1) and y to those of (2) lead to

$$\begin{aligned} xyz &= \lambda(xy + 2zx), \\ xyz &= \lambda(xy + 2yz). \end{aligned}$$

Subtracting one from the other gives us

$$0 = \lambda(2zx - 2yz) = 2\lambda z(x - y).$$

Hence, there are three possibilities, $\lambda = 0$ or $z = 0$ or $x = y$. If $\lambda = 0$, then inserting it to (1)–(3) leads to $yz = zx = xy = 0$, thus $0 + 2 \times 0 + 2 \times 0 - 12 = 0$, which leads to $-12 = 0$, which is impossible. Thus, $\lambda \neq 0$. Also, $z = 0$ cannot happen since the height of the box must be positive. Therefore, we must have $x = y$. Inserting $x = y$ to (3) leads to

$$x^2 = 4\lambda x,$$

which leads to $x = 4\lambda$ since $x \neq 0$. Plugging $x = 4\lambda$ into (1) gives us

$$4\lambda z = \lambda(4\lambda + 2z) \Longrightarrow z = 2\lambda.$$

Therefore, we have $x = y = 4\lambda$ and $z = 2\lambda$. Finally, inserting these to the constraint equation (4) leads to

$$16\lambda^2 + 4 \cdot 2\lambda \cdot 4\lambda - 12 = 0 \Longrightarrow 4\lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm\frac{1}{2}.$$

But we must have $x > 0$, $y > 0$, $z > 0$, to be a box. Thus, we must have $\lambda = 1/2$. This leads to $x = y = 2$, $z = 1$.

Step 2: Since there is only one such point, there is no need to compare the function value at this point with that of the other point. Therefore the maximum volume of the box is

$$\boxed{f(2,2,1) = 2 \cdot 2 \cdot 1 = 4 \text{ m}^3.}$$

Note: If you want to make sure that this is the maximum volume, then you can check the volume for some other values of (x, y, z) satisfying the constraint (4). For example, $(x, y, z) = (1, 1, 11/4)$ is such a point. But it is clear that

$$f(1,1,11/4) = 11/4 = 2.75 < 4 = f(2,2,1).$$

You must be satisfied by now!