Problem 1 (15 pts) Does the following *sequence* converge or diverge as $n \to \infty$? Give reasons for your answer. If it converges, find the limit.

(a) (7 pts)

$$a_n = n^2 e^{-n}.$$

Answer: Let us define the function $f(x) = x^2 e^{-x}$ for all $x \geq 1$. If $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$. Now,

$$\begin{aligned}
\lim_{x\to\infty} x^2 e^{-x} &= \lim_{x\to\infty} \frac{x^2}{e^x} \\
&= \lim_{x\to\infty} \frac{2x}{e^x} \quad \text{by l'Hôpital's rule.} \\
&= \lim_{x\to\infty} \frac{2}{e^x} \quad \text{by l'Hôpital's rule again.} \\
&= 0.
\end{aligned}$$

Therefore, this sequence converges to the limit 0.

(b) (8 pts)

$$a_n = n \cos \frac{1}{n}.$$

[Hint: Consider how the graph of $\cos x$ behave near $x = 0$. You may also want to use the fact: $\cos 1 \approx 0.54$.]

Answer: It diverges. Notice that $\cos \frac{1}{n} > \cos 1 \approx 0.54$ for $n = 2, 3, \ldots$. Hence we have:

$$n \cos \frac{1}{n} > 0.54n \quad \text{for } n = 2, 3, \ldots.$$

Now, the righthand side tends to ∞ as $n \to \infty$. Therefore, the lefthand side must go to ∞, i.e.,

$$\lim_{n\to\infty} n \cos \frac{1}{n} = \infty.$$

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Problem 2 (20 pts) Does the following series absolutely converge, conditionally converge, or diverge? Give reasons for your answer.

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$$

[Hint: You may want to use the following formula for a particular value of x:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=\mathrm{e}^{x}\quad \forall x\in\mathbb{R}.$$

Also, you may want to use the fact: $\mathrm{e}^{-1}\approx 0.36788$.]

Answer: Let $a_n=\left(1-\frac{1}{n}\right)^{n^2}$. Note that $a_n\geq 0$ for every $n\in\mathbb{N}$. Hence, there is no need to check the absolute convergence nor the conditional convergence. Simply checking the usual convergence suffice. Now, we will use the Root Test.

$$\sqrt[n]{a_n} = \left(1-\frac{1}{n}\right)^{n}=\left(1+\frac{-1}{n}\right)^{n}\to \mathrm{e}^{-1}\approx 0.36788 \quad \text{as } n\to\infty.$$

Hence, the series $\sum_{n=1}^{\infty} a_n$ (absolutely) converges via the Root Test since $\rho=\mathrm{e}^{-1}<1$.

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Problem 3 (20 pts) Does the following series absolutely converge, conditionally converge, or diverge? Give reasons for your answer.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

[Hint: Use either the Comparison Test or the Integral Test.]

Answer (via the Comparison Test): Let $a_n = \dfrac{\ln n}{n}$. It is clear that $a_n \geq 0$ for every $n \in \mathbb{N}$. So, we can use the Comparison Test. Notice that $\ln n > 1$ for every $n \geq 3$ since $e \approx 2.718$. Therefore,

$$\frac{\ln n}{n} > \frac{1}{n} \quad \text{for every } n \geq 3.$$

The series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges because this is the harmonic series. Therefore, by the Comparison Test, the series $\displaystyle\sum_{n=3}^{\infty} a_n$ diverges so does $\displaystyle\sum_{n=1}^{\infty} a_n$.

Answer (via the Integral Test): Let $f(x) = \dfrac{\ln x}{x} \geq 0$ for $x \geq 1$. Then, $f(n) = \dfrac{\ln n}{n} \geq 0$ for every $n \in \mathbb{N}$. So, let us set $a_n = f(n)$. Hence, we can apply the Integral Test, i.e., $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\int_1^{\infty} f(x)\,\mathrm{d}x$ share the same fate. Now notice that using Integration by Parts, we have

$$\int \frac{\ln x}{x}\,\mathrm{d}x = (\ln x)^2 - \int \frac{\ln x}{x}\,\mathrm{d}x.$$

That is,

$$\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{1}{2}(\ln x)^2.$$

(You can check this is correct by differentiating the righthand side.) Hence,

$$\int_1^{\infty} \frac{\ln x}{x}\,\mathrm{d}x = \left[\frac{1}{2}(\ln x)^2\right]_1^{\infty} = \infty.$$

That is, this integral diverges so does this series via the Integral Test.

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Problem 4 (20 pts) Determine the radius and the interval of convergence of the power series:

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} n(x-1)^n.$$

Justify your answers.

Answer: Let $a_n = (-1)^{n-1} n(x-1)^n$. Then,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^n(n+1)(x-1)^{n+1}}{(-1)^{n-1}n(x-1)^n}\right| \\ &= \frac{n+1}{n}|x-1| \to |x-1| \quad \text{as } n \to \infty \text{ regardless of the value of } x. \end{aligned}$$

Therefore, if $|x-1| < 1$, then this power series converges *absolutely* (and hence converges) by the Ratio Test. This means that $\boxed{\text{the radius of convergence is } R = 1.}$

As for the interval of convergence, we need to check the end points of the obvious interval $-1 < x-1 < 1$, i.e., $0 < x < 2$. If $x = 0$, then $f(0) = \sum_{n=1}^{\infty} (-1)^{2n-1} n = -\sum_{n=1}^{\infty} n$. The nth term of the series does not approach zero therefore the series diverges, specifically to $-\infty$. Hence, $x = 0$ cannot be included in the interval of convergence. For $x = 2$, $f(2) = \sum_{n=1}^{\infty} (-1)^{n-1} n$, which diverges because the nth term of the series does not approach zero. Hence, $x = 2$ cannot be included in the interval of convergence either. Therefore, $\boxed{\text{the interval of convergence is } 0 < x < 2, \text{ or } x \in (0,2).}$

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Problem 5 (25 pts) Let $f(x) = \cos x$.

(a) (10 pts) Find the Maclaurin series for $f(x)$.

Answer: Let $f(x) = \cos x$. Then, we have the following derivatives: $f^{(2k)}(x) = (-1)^k \cos x$, $f^{(2k+1)}(x) = (-1)^{k+1} \sin x$, $k = 0, 1, \ldots$ Hence, $f^{(2k)}(0) = (-1)^k$ while $f^{(2k+1)}(0) = 0$. Therefore, we have the following Taylor series of $\cos x$ at $x = 0$:

$$\boxed{\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.}$$

(b) (10 pts) Suppose we want to approximate $\cos x$ using $P_2(x)$, i.e., the Taylor polynomial of order 2, centered at $x = 0$. Use the Remainder Estimation Theorem to determine the range of x if we want to keep the magnitude of error between $\cos x$ and $P_2(x)$ less than 0.0001, i.e., $|\cos x - P_2(x)| < 0.0001$.
[Hint: You may want to use the fact: $(0.0006)^{1/3} \approx 0.0843$.]

Answer: First of all, the Taylor polynomial of order 2 is clearly

$$P_2(x) = 1 - \frac{x^2}{2!}.$$

Using Taylor's formula, we have

$$\cos x = P_2(x) + R_2(x) = 1 - \frac{x^2}{2!} + \frac{f^{(3)}(c)}{3!} x^3 \quad \text{for some } c \text{ between 0 and } x.$$

Since $|f^{(3)}(c)| = |\sin c| \leq 1$ for all values of c. Thus by the Remainder Estimation Theorem

$$|\cos x - P_2(x)| = |R_2(x)| \leq \frac{1}{3!}|x|^3.$$

Now to determine the range of x values for which the magnitude of error between $\cos x$ and $P_2(x)$ less than 0.0001, we find x such that

$$|R_2(x)| \leq \frac{1}{3!}|x|^3 < 0.0001 \Longleftrightarrow \frac{1}{3!}|x|^3 < 0.0001 \Longleftrightarrow |x| < (0.0006)^{1/3} \approx 0.0843.$$

Hence, the desired range of x is $\boxed{-0.0843 < x < 0.0843}$.

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(c) (5 pts) Prove $\cos 2\theta = \cos^2\theta - \sin^2\theta$ using Euler's Identity.

Answer: In Euler's Identity, we substitute 2θ for θ to get

$$e^{i2\theta} = \cos 2\theta + i\sin 2\theta. \tag{1}$$

On the other hand,

$$\begin{aligned} e^{i2\theta} &= \left(e^{i\theta}\right)^2 \qquad (2)\\ &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta + 2\cos\theta \cdot i\sin\theta + i^2\sin^2\theta \\ &= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta) \end{aligned}$$

Comparing the real part of Equations (1) and (2), we have:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Note also that comparing the imaginary part of Equations (1) and (2), we can also derive:

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

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