MAT 271: Applied & Computational Harmonic Analysis
Supplementary Notes IV (Revised as of 2/9/10) by Naoki Saito

The Discrete Fourier Transform (DFT)

- The DFT can be viewed as either an approximation to the Fourier transform or an approximation to the Fourier series coefficients.
- Suppose $f \in L^2[-A/2, A/2]$, and $f(x) = 0$ for $|x| > A/2$. That is, f is a space-limited, square integrable function, which is a reasonable assumption in practice. Then, we can invoke the dual version of the Shannon-Whittaker sampling theorem in the frequency domain, which can also be stated in terms of Fourier coefficients of the Fourier series (Recall that the Fourier transform of the the periodic functions gives the line spectrum in the frequency domain). In fact, we have the following relationship:

$$\hat{f}(k/A) = \int_{-A/2}^{A/2} f(x)\mathrm{e}^{-2\pi \mathrm{i} kx/A}\,\mathrm{d}x = \left\langle f, \mathrm{e}^{2\pi\mathrm{i}k\cdot/A}\right\rangle = \sqrt{A}\alpha_k, \quad k \in \mathbb{Z}. \tag{1}$$

- In general, $f \in L^2[-A/2, A/2]$ is not a band-limited function. Therefore, to have a finite length vector representing the frequency samples (or the Fourier coefficients), we need to truncate the frequency information for $|\xi| > \Omega/2$ for some $\Omega > 0$. This is the *first* source of error of DFT approximation to FT/FS. This truncation allows us to consider only k with $|k| \le A\Omega/2$.
- We now need to approximate the Fourier integration in (1) numerically. We use the *trapezoid rule*. Here is the *second* source of the error of DFT. Let's divide the interval $[-A/2, A/2]$ into N (*positive even integer*[1]) subintervals of equal length of $\Delta x = A/N$. Let $x_\ell = \ell\Delta x$, $\ell = (-N/2) : (N/2)$ be the points used in the trapezoid rule. Let $g(x) = f(x)\mathrm{e}^{-2\pi\mathrm{i}kx/A}$. Then we have

$$\hat{f}(k/A) \approx \frac{\Delta x}{2}\left\{ g(-A/2) + 2\sum_{\ell=-N/2+1}^{N/2-1} g(x_\ell) + g(A/2)\right\}.$$

  If we assume $f(-A/2) = f(A/2)$ (which we should do if possible by windowing or zero-padding), then the above approximation is simplified:

$$\hat{f}(k/A) \approx \Delta x \sum_{\ell=-N/2+1}^{N/2} g(x_\ell) = \frac{A}{N}\sum_{\ell=-N/2+1}^{N/2} f(\ell A/N)\,\mathrm{e}^{-2\pi\mathrm{i}k\ell/N},$$

- Now, let $f_\ell = f(\ell A/N)$ Then, the N-point DFT is defined as follows:

$$F_k \triangleq \frac{1}{\sqrt{N}}\sum_{\ell=-N/2+1}^{N/2} f_\ell\,\mathrm{e}^{-2\pi\mathrm{i}k\ell/N}, \quad k = -N/2+1, \ldots, N/2. \tag{2}$$

[1] All the subsequent matrix representations, etc. assumes this in this note. See [2, Sec. 3.1] for N being positive odd integer as well as the other cases, e.g., different starting and ending indices.

The factor $1/\sqrt{N}$ is to make DFT a unitary transformation (i.e., ℓ^2-norm (energy) preserving transformation, so that the Parseval & Plancherel equalities holds.)[2]

We now have the following relationship.

$$\hat{f}(k/A) = \sqrt{A}\alpha_k \approx \frac{A}{\sqrt{N}} F_k.$$

The N-point inverse DFT is defined, as you can imagine, as follows.

$$f_\ell \triangleq \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} F_k \, \mathrm{e}^{2\pi \mathrm{i} k\ell/N}, \quad \ell = -N/2+1, \ldots, N/2.$$

The proof of this formula gets easier when we use the vector-matrix notation later in this note.

- **[The reciprocity relations]** Let $\Delta\xi$ be a sampling rate in the frequency domain, i.e., $\Delta\xi = 1/A$. Since we know $\Delta x = A/N$, and $k/A = \Omega/2$ at $k = N/2$ (highest frequency in consideration), we have the following fundamental relations:

  $$\Delta x \Delta\xi = \frac{1}{N}, \quad A\Omega = N.$$

  Interpretation of these relations is very important. For example, fix N. Then increasing the length L implies increasing Δx, decreasing Ω, and decreasing $\Delta\xi$ (finer frequency sampling, but the frequency bandwidth also decreases). If we fix A, then increasing N (finer space sampling) implies decreasing Δx and increasing Ω while $\Delta\xi$ is kept constant (increasing the frequency bandwidth).

- **[A vector-matrix notation of DFT]** We can gain great insights by expressing DFT using vector-matrix notation. To do this, we need to define a couple of things. Let $\omega_N \triangleq \mathrm{e}^{2\pi i/N}$, i.e., Nth root of unity. Note that $\overline{\omega}_N = \omega_N^{-1}$, $\omega_N^0 = \omega_N^N = 1$, $\omega_N^{N/2} = -1$, and $\omega_N^{k+N} = \omega_N^k$ for any $k \in \mathbb{Z}$. Then, define a column vector

  $$\boldsymbol{w}_N^k \triangleq \frac{1}{\sqrt{N}} \left(\omega_N^{k\cdot 0}, \omega_N^{k\cdot 1}, \ldots, \omega_N^{k\cdot N/2}, \ldots, \omega_N^{k\cdot(N-1)} \right)^T, \quad k = 0, \ldots, N-1.$$

  We also define another column vector

  $$\widetilde{\boldsymbol{w}}_N^k \triangleq \frac{1}{\sqrt{N}} \left(\omega_N^{k\cdot(-N/2+1)}, \omega_N^{k\cdot(-N/2+2)}, \ldots, \omega_N^{k\cdot 0}, \ldots, \omega_N^{k\cdot N/2} \right)^T, \quad k = -N/2+1, \ldots, N/2.$$

  Using the properties of ω_N listed above, one can easily show that

  $$\widetilde{\boldsymbol{w}}_N^k = P_N \boldsymbol{w}_N^k, \quad P_N \triangleq T_N^{-1} S_N,$$

  where T_N shifts vector entries circularly in one step "down", i.e., $T_N(a_1, \ldots, a_N)^T = (a_N, a_1, \ldots, a_{N-1})^T$, and its matrix representation is

  $$T_N \triangleq \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

[2] Note that the definition used in the standard book [2] uses the factor $1/N$ instead, which makes DFT non-unitary. You need to be careful about the definition of DFT when you read literature and use software packages. More about this in the end of this note.

Note that $T_N^{-1} = T_N^T$ represents the "up" circular shift operation. In MATLAB, $T_N^m \boldsymbol{a}$ corresponds to `circshift(a,m)` where m is an arbitrary integer (positive or negative). S_N is equivalent to `fftshift` in MATLAB, so its matrix representation is

$$S_N \triangleq \begin{bmatrix} O_{N/2} & I_{N/2} \\ I_{N/2} & O_{N/2} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that $S_N^T = S_N^{-1} = S_N$. Now, just in case, the matrix representation of P_N is:

$$P_N \triangleq \left[\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \end{array}\right].$$

Let $\boldsymbol{f} = (f_{-N/2+1}, \ldots, f_{N/2})^T$ be a vector of sampled points of $f(x)$ in x. Now DFT can be written as follows:

$$F_k = \left\langle \boldsymbol{f}, \widetilde{\boldsymbol{w}}_N^k \right\rangle, \quad k = -N/2+1, \ldots, N/2.$$

Finally, define an N-point DFT matrix commonly used in the literature

$$W_N \triangleq \left[\boldsymbol{w}_N^0 \;\middle|\; \boldsymbol{w}_N^1 \;\middle|\; \cdots \;\middle|\; \boldsymbol{w}_N^{N-1} \right]$$

whereas we define the following matrix compliant with our definition of DFT (2):

$$\widetilde{W}_N \triangleq \left[\widetilde{\boldsymbol{w}}_N^{-N/2+1} \;\middle|\; \widetilde{\boldsymbol{w}}_N^{-N/2+2} \;\middle|\; \cdots \;\middle|\; \widetilde{\boldsymbol{w}}_N^{N/2} \right] = P_N W_N P_N^T.$$

Let $\boldsymbol{F} = (F_{-N/2+1}, \ldots, F_{N/2})^T \in \mathbb{C}^N$. Then, the N-point DFT/IDFT can be conveniently written as:

$$\boldsymbol{F} = \widetilde{W}_N^* \boldsymbol{f}, \quad \boldsymbol{f} = \widetilde{W}_N \boldsymbol{F},$$

where $\widetilde{W}_N^*$ is an hermitian conjugate (transposition followed by element-wise complex conjugation) of $\widetilde{W}_N$, and also often written as $\widetilde{W}_N^H$ in literature. In fact, $\widetilde{W}_N^* = \left(P_N W_N P_N^T\right)^* = P_N W_N^* P_N^T$. We also denote $\mathcal{F}_N[\boldsymbol{f}] \triangleq \widetilde{W}_N^* \boldsymbol{f}$.

- **[Theorem 1]** Both W_N and $\widetilde{W}_N$ are N-by-N unitary matrix. In other words, both $\{\boldsymbol{w}_N^k\}_{k=0}^{N-1}$ and $\{\widetilde{\boldsymbol{w}}_N^k\}_{k=-N/2+1}^{N/2}$ are orthonormal bases of $\mathbb{C}^N$.
(Proof) Exercise. A main thing is to prove $\left\langle \boldsymbol{w}_N^k, \boldsymbol{w}_N^\ell \right\rangle = \delta_{k,\ell}$.

- **[Theorem 2]** All the eigenvalues of W_N and $\widetilde{W}_N$ are $1, -1, \mathrm{i}, -\mathrm{i}$.
(Proof) See [1] or [2]. Note that from this theorem we have $W_N^4 = \widetilde{W}_N^4 = I_N$.

- **[Pictorial view of the matrix W_N^*]**. Using the properties of ω_N, in particular the periodicity with period N, we have

$$W_N^* = \begin{bmatrix} (\boldsymbol{w}_N^0)^* \\ (\boldsymbol{w}_N^1)^* \\ (\boldsymbol{w}_N^2)^* \\ \vdots \\ (\boldsymbol{w}_N^{N/2})^* \\ \vdots \\ (\boldsymbol{w}_N^{N-1})^* \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \overline{\omega}_N^1 & \overline{\omega}_N^2 & \dots & \overline{\omega}_N^{N-1} \\ 1 & \overline{\omega}_N^2 & \overline{\omega}_N^4 & \dots & \overline{\omega}_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \overline{\omega}_N^{N/2} & \overline{\omega}_N^{2N/2} & \dots & \overline{\omega}_N^{(N-1)N/2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \overline{\omega}_N^{N-1} & \overline{\omega}_N^{2(N-1)} & \dots & \overline{\omega}_N^{(N-1)(N-1)} \end{bmatrix}$$

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \dots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \dots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega_N^{-N/2+1} & \omega_N^{2(-N/2+1)} & \dots & \omega_N^{(N-1)(-N/2+1)} \\ 1 & \omega_N^{-N/2} & \omega_N^{-2N/2} & \dots & \omega_N^{-(N-1)N/2} \\ 1 & \omega_N^{N/2-1} & \omega_N^{2(N/2-1)} & \dots & \omega_N^{(N-1)(N/2-1)} \\ 1 & \omega_N^{N/2-2} & \omega_N^{2(N/2-2)} & \dots & \omega_N^{(N-1)(N/2-2)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega_N^1 & \omega_N^2 & \dots & \omega_N^{N-1} \end{bmatrix}.$$

The following figure shows the matrix W_N^* and $\widetilde{W}_N^*$ with $N = 16$ as waveforms. Note that the first row vector of a matrix is displayed in the bottom while the last row in the top in each figure. Note the change of the locations of the basis vectors as well as symmetry $(W_N^*)^T = W_N^*$, $(\widetilde{W}_N^*)^T = \widetilde{W}_N^*$.

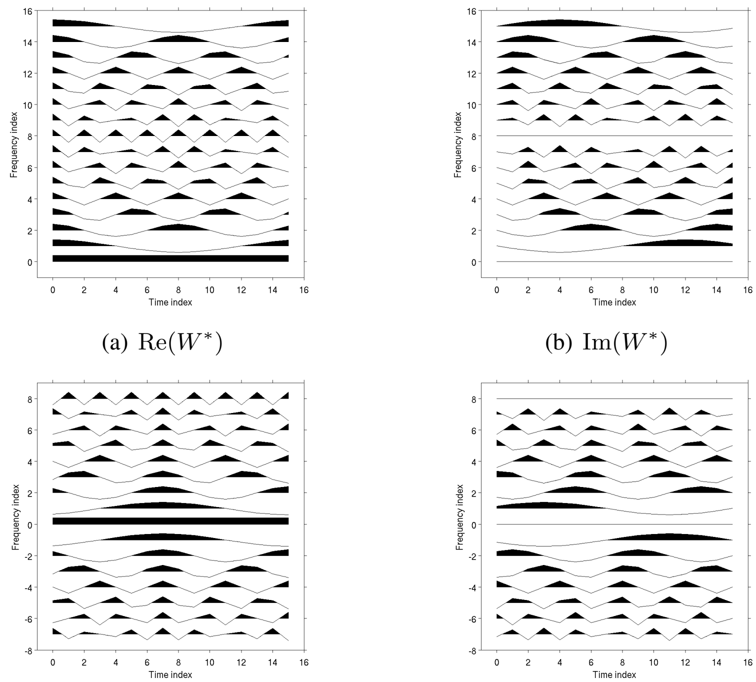
- **[Different forms of DFT]** It is amazing to know that the definition of DFT varies with the software systems one uses. We should always be careful what is the exact definition of the DFT for each software system.

MATLAB, R, S-Plus: $F_k = \sum_{\ell=1}^{N} f_\ell \, \mathrm{e}^{-2\pi\mathrm{i}(k-1)(\ell-1)/N}$ for $k = 1 : N$.

Mathematica: $F_k = \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} f_\ell \, \mathrm{e}^{2\pi\mathrm{i}(k-1)(\ell-1)/N}$ for $k = 1 : N$.

Maple: $F_{k+1} = \frac{1}{N} \sum_{\ell=0}^{N-1} f_\ell \, \mathrm{e}^{-2\pi\mathrm{i}k\ell/N}$ for $k = 0 : (N-1)$.

MathCad: $F_k = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} f_\ell \, \mathrm{e}^{2\pi\mathrm{i}k\ell/N}$ for $k = 0 : (N-1)$.



(a) $\mathrm{Re}(W^*)$ (b) $\mathrm{Im}(W^*)$

(c) $\mathrm{Re}(\widetilde{W}^*)$ (d) $\mathrm{Im}((\widetilde{W}^*)$

Hence, the DFT we defined in this note, i.e., $\boldsymbol{F} = \widetilde{W}_N^* \boldsymbol{f}$, can be realized by the following MATLAB command (assuming that `f` is a 1D vector):

```
F=circshift(fftshift(fft(fftshift(circshift(f,1)))),-1)/sqrt(length(f));
```

For more information about the DFT including higher-dimensional versions, see [2]. Also, the DFT matrix has more *profound* properties. See the challenging and deep paper by [1].

References

[1] L. AUSLANDER AND R. TOLIMIERI, *Is computing with the finite Fourier transform pure or applied mathematics?*, Bull. Amer. Math. Soc., 1 (1979), pp. 847–897.

[2] W. L. BRIGGS AND V. E. HENSON, *The DFT: An Owner's Manual for the Discrete Fourier Transform*, SIAM, Philadelphia, PA, 1995.