On the amplitude- and frequency-modulation decomposition of signals[a)]

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In general, the problem of determining the amplitude and frequency modulations (AM and FM) of a signal is ill posed because there is an unlimited number of combinations of AM and FM that will generate a given signal. Although Gabor proposed a method for uniquely defining the AM and FM of a signal, namely via the analytic signal, the results obtained are sometimes physically paradoxical. In this paper, four reasonable physical conditions that the calculated AM and FM of a signal should satisfy are proposed. The analytic signal method generally fails to satisfy two of the four conditions. A method utilizing the positive (Cohen–Posch) time-frequency distribution and time-varying coherent demodulation of the signal is given for obtaining an AM and FM that satisfy the four proposed conditions. Contrary to the accepted definition, the instantaneous frequency (i.e., the FM) that satisfies these conditions is generally not the derivative of the phase of the signal. Rather, the phase is separated into two parts, one which gives the instantaneous frequency via differentiation, and the other which can be interpreted either as phase modulation or quadrature amplitude modulation of the signal. Examples are given for synthetic signals and speech, with comparisons to the analytic signal method. © *1996 Acoustical Society of America.*

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INTRODUCTION

The transmission of information in many natural and man-made systems is accomplished by varying, or modulating, the amplitude and/or frequency of a signal. Determining the inherent amplitude and frequency modulations (AM and FM) of a signal, however, is a challenging problem with a long history.[1–10] Typically, the signal is modeled as the real part of a complex signal with amplitude $A(t)$ and phase $\varphi(t)$,

$$x(t)=\Re[A(t)e^{j\varphi(t)}]. \tag{1}$$

Accordingly, the AM is taken to be $A(t)$ (or its magnitude, $|A(t)|$, which is also called the signal envelope), and the FM is taken as $d\varphi(t)/dt=\dot{\varphi}(t)$, which is the ''instantaneous frequency'' of the signal.

Note, however, that there is an unlimited number of combinations of AM and FM—i.e., $A(t)$ and $\varphi(t)$—that could generate a given signal, and therefore Eq. (1) is not a unique representation. To illustrate, Fig. 1 shows a simple two-tone signal and two possible AM–FM pairs, both of which yield the signal via the equation above.

Because Eq. (1) is not a unique representation, we must rely upon physical (and possibly other) considerations to guide us in making a reasonable choice for $A(t)$ and $\varphi(t)$. For example, in the two-tone case, the signal is bounded, and its spectrum is bandlimited. Hence, it is physically reasonable to require that the AM be bounded, and the FM be limited to the same spectral band as the signal. AM–FM pairs that do not satisfy these physical conditions (such as those in Fig. 1), while mathematically correct per Eq. (1), are physically paradoxical.

In this paper, we consider the determination of the AM and FM of a signal. Building on the approach of Vakman,[8] we propose physical conditions that the calculated AM and FM should satisfy. We show that, in general, taking the FM, or instantaneous frequency, as the derivative of the phase violates two of the conditions. We then present a method, utilizing positive time-frequency distributions (TFDs)[11,12] and time-varying coherent demodulation, for determining an FM and an AM that satisfy the proposed conditions.

I. PHYSICAL CONDITIONS FOR THE AM AND FM OF A SIGNAL

Letting $A(t)$ denote the calculated AM and $\omega(t)$ denote the calculated FM of a signal, we propose the following four physical conditions that these quantities should satisfy:

1. If the signal is bounded in magnitude, then the magnitude of the AM should be bounded:

$$|x(t)|<\infty \Rightarrow |A(t)|<\infty. \tag{2}$$

2. If the signal is limited in frequency range (i.e., its spectrum is zero outside some range of frequencies $\omega_l<\omega<\omega_u$), then the FM should likewise be limited to the same range:

$$|X(\omega)|^2=0, \quad \omega\notin(\omega_l<\omega<\omega_u) \Rightarrow \omega_l<\omega(t)<\omega_u. \tag{3}$$

3. If the signal is of constant amplitude and constant frequency, i.e., if $x(t)=A_0\cos(\omega_0 t+\phi_0)$ where ϕ_0 is an arbitrary phase constant, then the magnitude of the AM should equal the magnitude of A_0 and the FM should equal ω_0:

$$A_0\cos(\omega_0 t+\phi_0) \Rightarrow |A(t)|=|A_0|, \quad \omega(t)=\omega_0. \tag{4}$$

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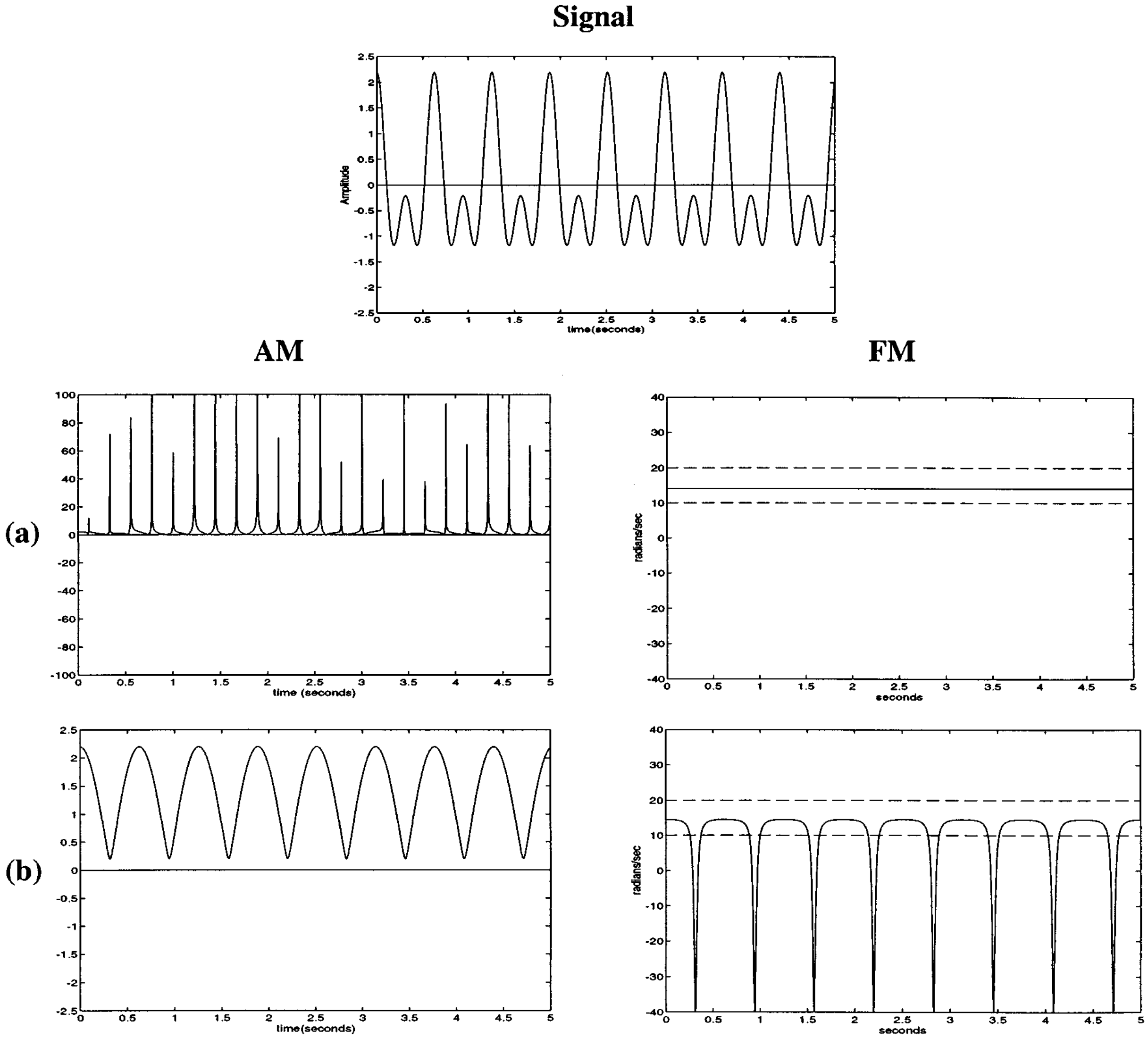

FIG. 1. Two possible AM-FM candidates [(a) and (b)] for a two-tone signal (top). The dashed lines in the FM plots indicate the individual frequencies of the two-tone signal. Given that the signal was bounded in amplitude and frequency range, neither AM-FM pair is physically reasonable, although both satisfy Eq. (1).

4. If the signal is scaled in amplitude by a constant c, then the AM should be scaled by the same constant, and the FM should be unaffected:

$$\text{if } x(t) \Rightarrow A(t), \omega(t),$$

$$\text{then } cx(t) \Rightarrow cA(t), \omega(t). \tag{5}$$

The first condition above is more restrictive than Vakman's ''amplitude continuity'' constraint, and the second condition is not considered by Vakman.[8] The last two conditions are essentially equivalent to Vakman's ''harmonic correspondence'' and ''phase independence of scaling'' conditions, respectively. Any method that fails to satisfy any one of Vakman's conditions necessarily fails to satisfy at least one of ours. These methods include, among others, the Teager–Kaiser method, Mandelstam's method and Shekel's method.[18]

II. THE ANALYTIC SIGNAL AND TIME-FREQUENCY DISTRIBUTIONS

Gabor proposed a method for unambiguously defining the amplitude and phase by generating a specific complex signal (the analytic signal) from the given real signal, via the Hilbert transform.[2] Letting $x(t)$ and $y(t)$ denote, respectively, the real and imaginary parts of the analytic signal, the phase and amplitude are calculated in the usual way

$$\varphi(t) = \operatorname{atan}(y(t)/x(t)) \tag{6}$$

$$A(t) = (x(t) + jy(t))e^{-j\varphi(t)}. \tag{7}$$

The instantaneous frequency and signal envelope are given by $\dot{\varphi}(t)$ and $|A(t)| = \sqrt{x^2(t) + y^2(t)}$, respectively.

In the time-frequency literature, instantaneous frequency is interpreted as the average frequency at each time because, for an unlimited number of time-frequency distributions

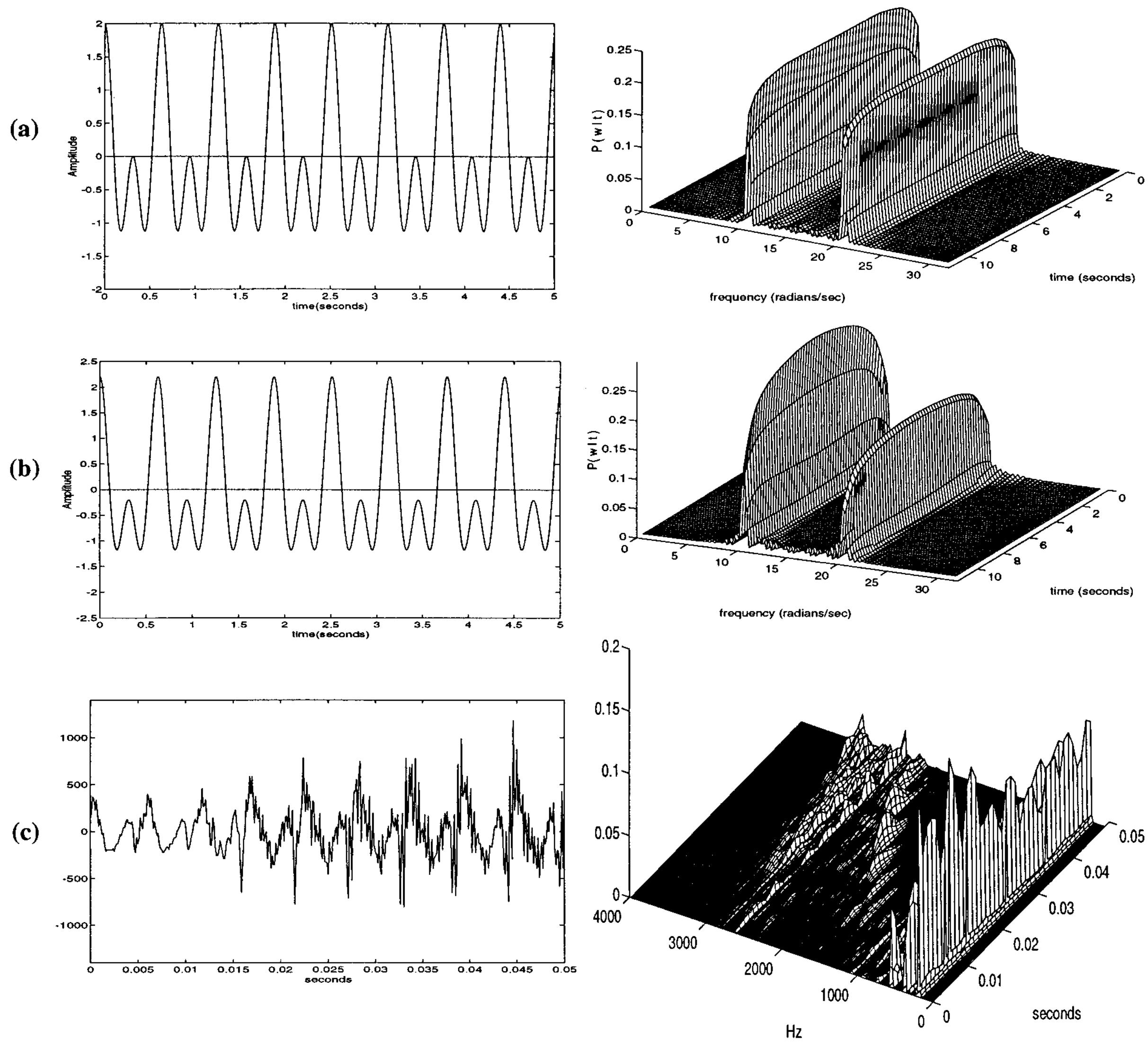

FIG. 2. Time waveforms (left) and time-conditional positive (Cohen–Posch) distributions $P(\omega|t)$ (right) of (a) two tones of equal strength; (b) two tones of unequal strength; and (c) voiced speech. The conditional mean frequency [Eq. (10)] gives the average frequency at each time, which is the interpretation of ''instantaneous frequency'' in the time-frequency literature. The distributions were computed per Ref. 12.

(TFDs) $P(t,\omega)$ of the signal $A(t)e^{j\varphi(t)}$, the first conditional moment in frequency [integrals span $(-\infty,\infty)$ unless noted otherwise]

$$\langle\omega\rangle_t=\int \omega P(\omega|t)d\omega = \int \omega P(t,\omega)d\omega \bigg/ \int P(t,\omega)d\omega, \tag{8}$$

which gives the average frequency at each time, equals the derivative of the phase:[10,13,14]

$$\langle\omega\rangle_t=\dot{\varphi}(t). \tag{9}$$

This equality, however, holds for *any* complex signal, not just the analytic signal.[13] [TFDs that satisfy Eq. (9) for any complex signal $A(t)e^{j\varphi(t)}$ are not positive (i.e., they contain negative values),[13] a result that is difficult to reconcile with the desired interpretation of such TFDs as joint energy density functions.] Thus Eq. (9) offers no more guidance in the selection of a particular phase and amplitude than does Eq. (1); therefore, the analytic signal is frequently used in computing the TFD for agreement with Gabor's definition of instantaneous frequency.

In light of the proposed conditions, however, there is a problem: it is well known that the derivative of the phase of a complex signal (analytic or otherwise) can extend beyond the frequency range of the signal[3,15] thereby violating the second condition. Indeed, for the analytic signal, the first and second conditions are violated. [The first condition can be violated by the analytic signal because the Hilbert transform is not bounded-input/bounded-output stable. The analytic signal will be unbounded in magnitude wherever there is a finite discontinuity in the real, bounded signal. A sufficient condition for a bounded real signal $x(t)$ with Fourier transform $X(\omega)$ to have a bounded analytic signal is that $|X(\omega)|$ be integrable, which follows from $\int|X(\omega)|d\omega \geqslant 2|\int_0^\infty X(\omega)e^{j\omega t}d\omega|$.]

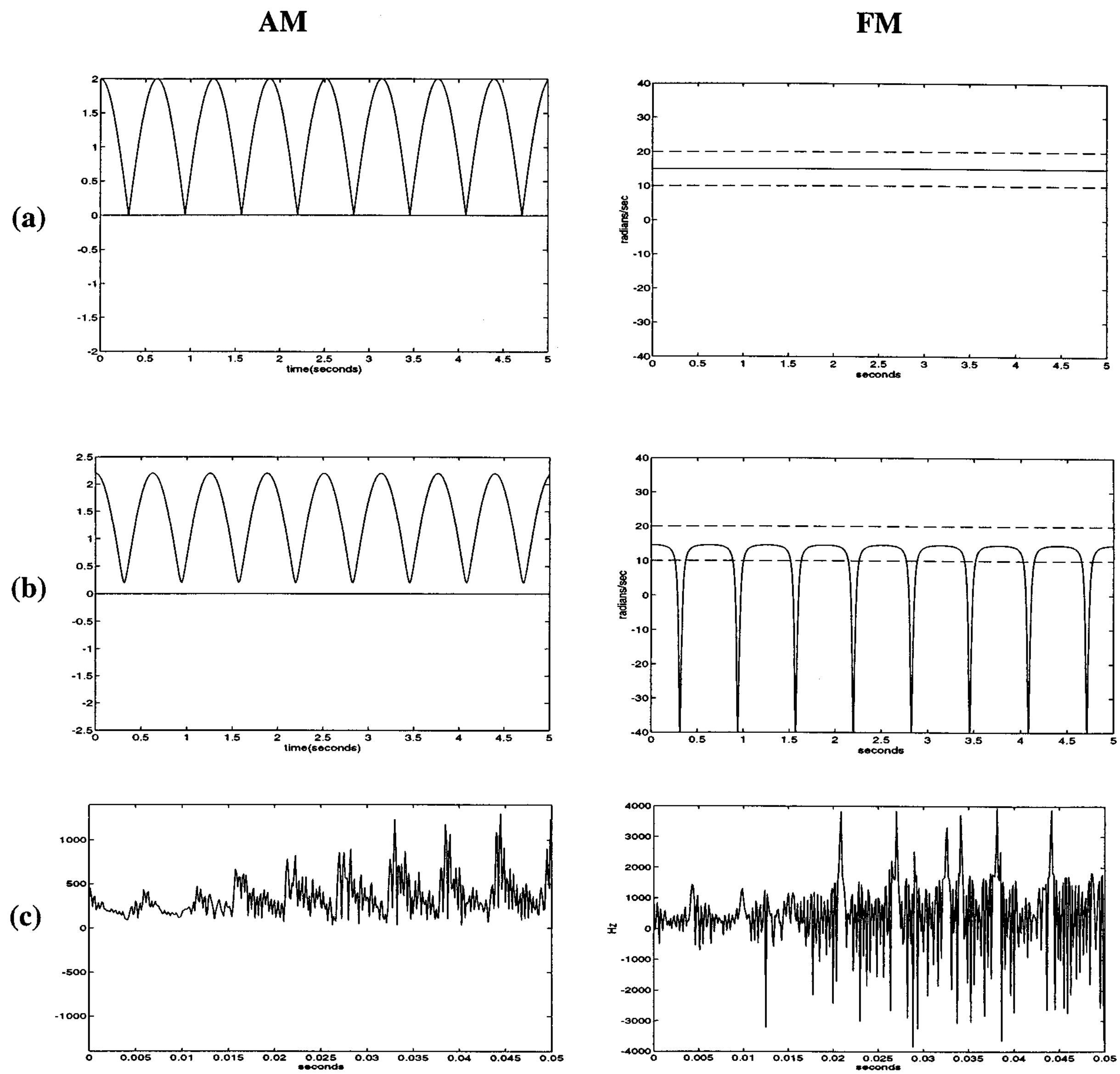

FIG. 3. AM and FM via the analytic signal method for the three signals in Fig. 2. (a) Two tones of equal strength; (b) two tones of unequal strength; (c) voiced speech. Dashed lines in the FM plots in (a) and (b) delineate the frequencies of the two tones. The analytic signal approach generally violates the second condition (FM limited to the same frequency range as the signal), and it can violate the bounded AM condition (although that condition is met for these signals). Note that the FM, or instantaneous frequency taken as the derivative of the phase, equals the mean of the two frequencies in (a), consistent with the interpretation of instantaneous frequency as ''the average frequency at each time.'' However, that is not the case in (b) or (c). The only case for which the derivative of the phase of the two-tone signal does not extend beyond the tones at ω_0 and ω_1 is when the tones are of equal strength,[16] as in (a). In (c), the FM exceeds the bandwidth of the signal, and indicates extremely rapid frequency changes that are physically impossible in voiced speech, given the inertia of the system (e.g., mass of the tongue).

III. A METHOD FOR FINDING AN AM-FM PAIR SATISFYING THE PHYSICAL CONDITIONS

To find an AM-FM pair that satisfies the desired physical conditions, we turn to the theory of positive (Cohen–Posch) TFDs,[11,12] which satisfy Eq. (9) for some complex signals, but not all.[16] Accordingly, the positive TFDs eliminate many possibilities for the AM and FM and can therefore potentially guide us in the selection of a pair consistent with the conditions.

We propose that the FM, $\omega(t)$, of a real signal be calculated from a positive TFD of the signal as

$$\omega(t)=\langle\omega\rangle_t=2\int_0^\infty \omega P(\omega|t)\,d\omega, \tag{10}$$

where a one-sided integral is taken to obtain a result different from zero for real signals. [If the given signal happens to be complex, one can substitute $\omega(t)=\langle\omega\rangle_t=\int\omega P(\omega|t)\,d\omega$ for (10).] It is shown in the Appendix that this approach yields an FM that satisfies the proposed conditions.

The AM can be obtained, given the FM, via time-varying coherent demodulation. First, calculate a phase as

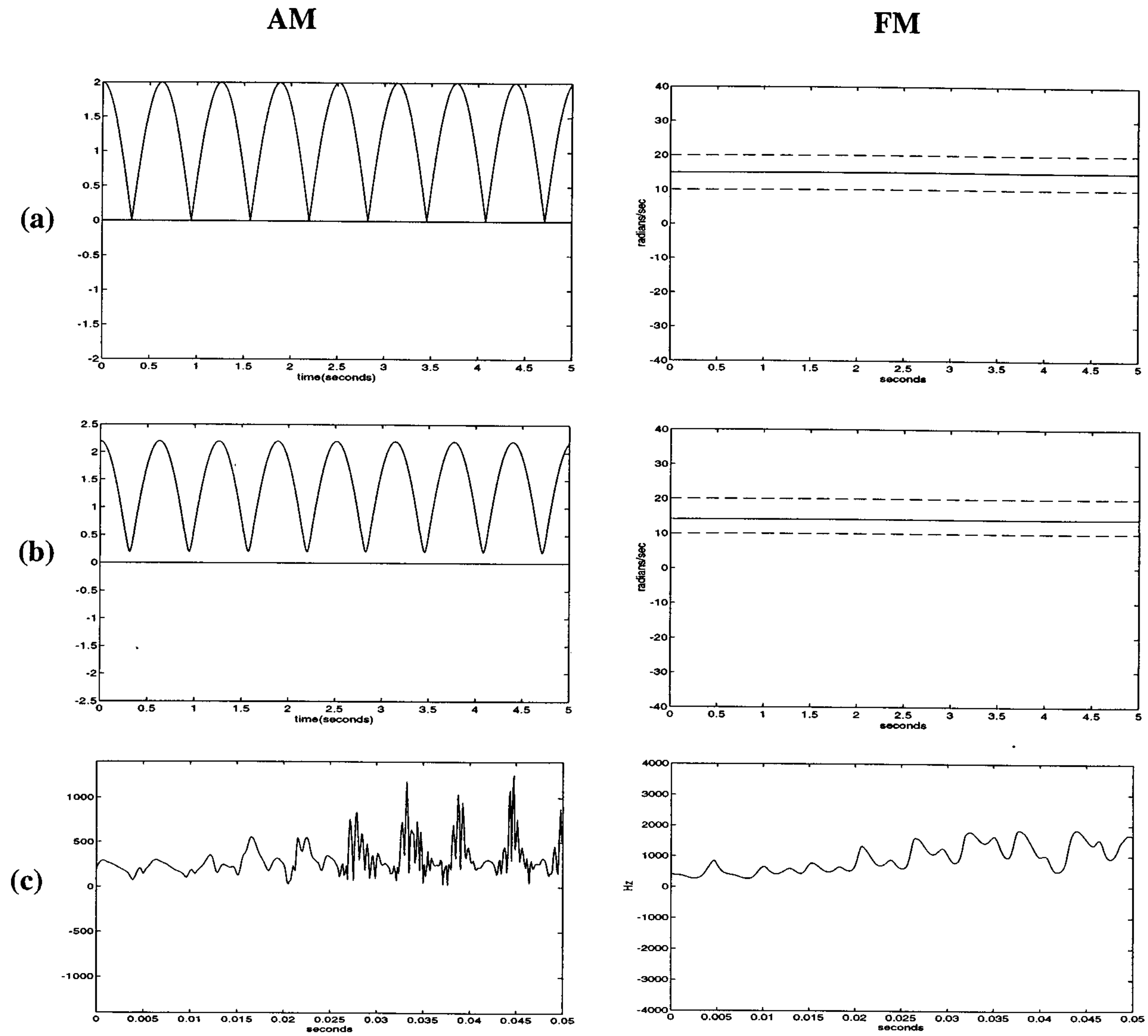

FIG. 4. AM and FM via the proposed method for the three signals in Fig. 2. (a) Two tones of equal strength; (b) two tones of unequal strength; (c) voiced speech. Dashed lines in the FM plots in (a) and (b) delineate the frequencies of the two tones. This approach satisfies the proposed conditions for the AM and FM of a signal. Furthermore, the instantaneous frequency, or FM, calculated as the conditional mean frequency of a positive TFD, is always interpretable as the average frequency at each time.

$$\varphi_F(t)=\int_{-\infty}^{t}\omega(\tau)\,d\tau. \tag{11}$$

Coherent demodulation is achieved by multiplying the real signal by the cosine and sine of the phase $\varphi_F(t)$, respectively, followed by time-varying low-pass filtering to obtain in-phase $A_I(t)$ and quadrature $A_Q(t)$ components of the AM, as follows:

$$A_I(t)=\int x(\tau)\cos(\varphi_F(\tau))h_{lp}(t,\tau)\,d\tau, \tag{12}$$

$$A_Q(t)=\int x(\tau)\sin(\varphi_F(\tau))h_{lp}(t,\tau)\,d\tau, \tag{13}$$

where $h_{lp}(t,\tau)$ is the time-varying impulse response of a low-pass filter with varying cutoff frequency $\langle\omega\rangle_t$ and a pass-band gain equal to two. This approach yields an AM $[\sqrt{A_I^2(t)+A_Q^2(t)}]$ that satisfies the proposed conditions (see the Appendix).

IV. EXAMPLES

We illustrate the proposed method and conditions for a two-tone signal and a speech signal. Comparison is made to the AM and FM obtained via the analytic signal. Discrete TFDs were computed via the method of Ref. 12, and the conditional mean frequency was computed as in Ref. 16. The time-varying filter used in the calculation of the AM per Eqs. (12), (13) was implemented as in Ref. 17.

Figure 2 shows the time waveforms and time-conditional positive distributions $P(\omega|t)$ for the signals considered. Figure 3 shows the AM and FM obtained for each signal via the analytic signal approach, and Fig. 4 shows the AM and FM obtained via the proposed method. For the two-tone signal $A_1e^{j\omega_1 t}+A_2e^{j\omega_2 t}$ (which is analytic if $\omega_1,\omega_2>0$) with $|A_1|=|A_2|$ [(a) in Figs. 2–5], the AM and FM obtained

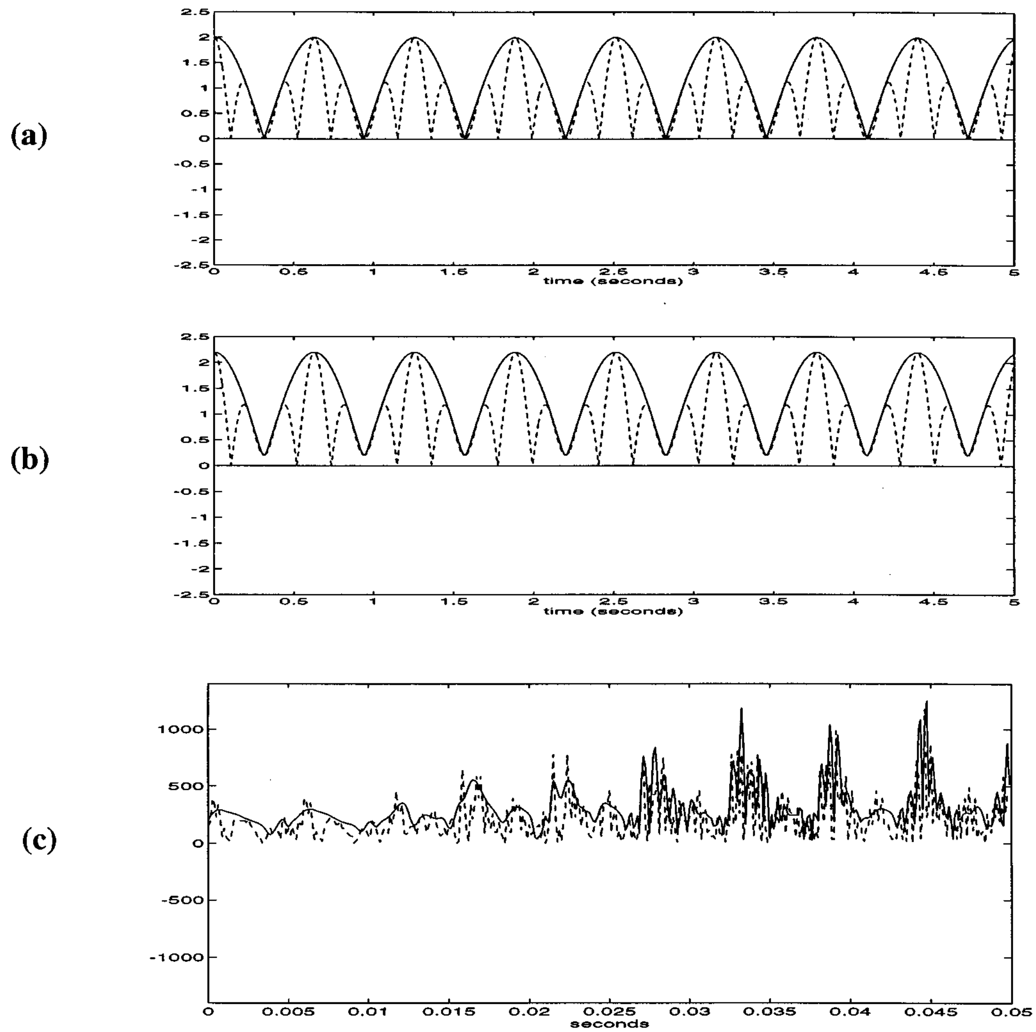

FIG. 5. Comparison of the signal magnitude (dashed) and the AM obtained via time-varying coherent demodulation (solid) for the three signals in Fig. 2: (a) two tones of equal strength, (b) two tones of unequal strength, and (c) speech. In all cases, the AM is a good match to the ''envelope'' of the signal.

is the same for both methods. For $|A_1| \neq |A_2|$ [(b) in Figs. 2–5], the AM ($|A(t)|$) is the same in both methods, but only the FM of the proposed approach satisfies the conditions and is consistent with the results for equal strength tones: it is a weighted average of the two tones. The derivative of the phase [FM in Fig. 3(b)], on the other hand, is highly erratic and extends beyond the frequency range of the signal; it cannot be interpreted as ''the average frequency at each time.''

The same is true for the derivative of the phase of the voiced speech signal [FM in Fig. 3(c)]. Any physical interpretation of this result, in terms of the mechanisms of speech production, is difficult to make. On the other hand, the conditional mean frequency of a positive TFD of the speech signal [Fig. 4(c)] exhibits general agreement with the rising formant observed in the TFD [Fig. 2(c)], and is indeed the average frequency at each time. Note the conditional mean frequency shows modulations occurring regularly with the pitch period, and within a single pitch period.

V. DISCUSSION

Coherent demodulation yields, in effect, a complex amplitude $A_I(t) + jA_Q(t) = A(t)e^{j\varphi_A(t)}$. Accordingly, the total signal phase is $\varphi(t)=\varphi_A(t)+\varphi_F(t)$, and per Eq. (1) we have

$$x(t)=\Re[A(t)e^{j(\varphi_A(t)+\varphi_F(t))}]$$

$$=A_I(t)\cos(\varphi_F(t))-A_Q(t)\sin(\varphi_F(t)) \tag{14a}$$

$$=A(t)\cos\left(\int_{-\infty}^{t}\omega(\tau)d\tau+\varphi_A(t)\right). \tag{14b}$$

The derivative of $\varphi_F(t)$ [i.e., $\omega(t)$] is the FM, while the phase $\varphi_A(t)$ can be interpreted in two different ways: per Eq. (14a), $\varphi_A(t)$ induces quadrature amplitude modulation, where $A_I(t)=A(t)\cos(\varphi_A(t))$ and $A_Q(t)=A(t)\sin(\varphi_A(t))$. Alternately, $\varphi_A(t)$ constitutes phase modulation (PM) per Eq. (14b).

To satisfy the bounded AM condition, the signal phase must be split into two parts. In other words, the phase $\varphi_F(t)$ obtained from the positive TFD cannot be taken as the total phase $\varphi(t)$ of the signal. To see this, consider the case where we take $\varphi_F(t)$ to be the total signal phase; solving Eq. (1) for the amplitude yields $A(t)=x(t)/\cos\varphi_F(t)$, which is generally unbounded [e.g., AM in Fig. 1(a)]. Taking $\varphi(t)=\varphi_A(t)+\varphi_F(t)$, on the other hand, yields an AM and FM that satisfy the proposed conditions.

VI. CONCLUSION

Determining the AM and FM of a signal is an ill-posed problem, in that there is an infinite number of amplitudes $A(t)$ and phases $\varphi(t)$ that will yield the given signal, per Eq. (1). While all of these pairs are mathematically legitimate, not all are physically sensible, in that they violate reasonable physical conditions.

We proposed four conditions that the calculated AM and FM of a signal should satisfy. We showed that the commonly accepted method for defining the AM and FM of a signal, namely as the amplitude and the derivative of the phase of the complex signal $A(t)e^{j\varphi(t)}$, generally violates some of the physical conditions. A method for calculating an AM and FM that satisfy the proposed conditions was presented. Examples were provided to illustrate the conditions and method, with comparisons to the analytic signal approach.

In general, for the conditions to be met, the phase must be separated into an FM part $\varphi_F(t)$ and an AM part $\varphi_A(t)$ (or equivalently, a PM part—see Sec. V). Positive TFDs allow one to determine $\varphi_F(t)$ via (11). Time-varying coherent demodulation can be used to determine the remaining phase $\varphi_A(t)$ [and the amplitude $A(t)$], via Eqs. (12) and (13).

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APPENDIX A: SATISFACTION OF CONDITIONS

1. Bounded AM

Time-varying coherent demodulation yields a bounded AM given a bounded signal provided that the filter is bounded-input/bounded-output stable, i.e., $\int|h_{lp}(t,\tau)|d\tau<\infty$. The sufficiency of this stability requirement follows from Eqs. (12) and (13) by using $\sqrt{A_I^2+A_Q^2} \leqslant \sqrt{A_I^2} + \sqrt{A_Q^2}$ and $|\int f(\tau)d\tau|\leqslant\int|f(\tau)|d\tau$. This requirement is not a limitation, and is readily met in practice.

2. Bandlimited FM

If the spectral density $|X(\omega)|^2$ is zero outside the band of frequencies $\omega_l<\omega<\omega_u$, then so is the time-conditional positive TFD $P(\omega|t)$.[11] Furthermore, because $P(\omega|t)$ is nonnegative, it follows that the conditional mean frequency $\langle\omega\rangle_t$ is limited to the same band, $\omega_l<\langle\omega\rangle_t<\omega_u$. Hence, condition 2 is satisfied. TFDs that are not positive generally fail this condition as can TFDs that are non-negative but fail to be zero outside the band of frequencies $\omega_l<\omega<\omega_u$.[14,18]

3. Constant AM and FM for pure tone

The maximum entropy positive TFD of a pure tone is the distribution $|x(t)|^2|X(\omega)|^2/E$ (where E is the signal energy),[12] which is also the correlationless distribution. The conditional mean frequency is therefore equal to the mean frequency here, i.e., $\langle\omega\rangle_t=\langle\omega\rangle=\omega_0$ (see Appendix B), and hence the FM is constant and equal to ω_0 as required.

Time-varying coherent demodulation yields a constant AM for this signal. First, note that because the corner frequency of the filter is constant, $\langle\omega\rangle_t=\omega_0$, the superposition integrals of Eqs. (12) and (13) become convolution integrals (i.e., the filter is linear time-invariant here). Evaluating the integrals for $x(t)=A_0\cos(\omega_0 t+\phi_0)$ yields, after applying trigonometric identities,

$$A_I(t)=\frac{1}{2}A_0\int(\cos(2\omega_0\tau+\phi_0)+\cos(\phi_0))h_{lp}(t-\tau)d\tau, \tag{A1}$$

$$A_Q(t)=\frac{1}{2}A_0\int(\sin(2\omega_0\tau+\phi_0)-\sin(\phi_0))h_{lp}(t-\tau)d\tau, \tag{A2}$$

[where without loss of generality we assume $\varphi_F(-\infty)=0$ in evaluating Eq. (11)]. The cosine and sine terms of frequency $2\omega_0$ are removed by the filter, while in the passband, the filter has a gain of two, so we have

$$A_I(t)=A_0\cos(\phi_0), \tag{A3}$$

$$A_Q(t)=-A_0\sin(\phi_0), \tag{A4}$$

from which it follows that the AM $\sqrt{A_I^2(t)+A_Q^2(t)}$ equals $|A_0|$, as desired.

4. Amplitude scaling affects only AM

If a signal is scaled in amplitude, $x(t)\to cx(t)$, its conditional mean frequency $\langle\omega\rangle_t$ is unaffected because of the normalization inherent in its calculation [see (10)—since $P(\omega|t)=P(t,\omega)/\int P(t,\omega)d\omega$, scale factors cancel]. Accordingly, $\varphi_F(t)$ and $h_{lp}(t,\tau)$ are also unaffected by amplitude scaling. It follows by inspection of Eqs. (12) and (13) that only the AM is affected, and it scales appropriately.

APPENDIX B: CONDITIONAL MEAN FREQUENCY OF A PURE TONE

For the real tone $A_0\cos(\omega_0 t+\phi_0)$ with positive TFD $P(t,\omega)=|x(t)|^2|X(\omega)|^2/E$, we have by Eq. (10),

$$\langle\omega\rangle_t=2\int_0^\infty\omega|X(\omega)|^2d\omega\Big/\int_{-\infty}^\infty|X(\omega)|^2d\omega=\langle\omega\rangle. \tag{B1}$$

Care must be exercised in the evaluation of (B1), because the signal is not finite energy. Accordingly, we will evaluate the conditional mean frequency for a finite-duration tone, and then take the limit as the duration goes to infinity. Consider, therefore, the signal

$$x(t)=e^{-t^2/2\sigma^2}A_0\cos(\omega_0 t+\phi_0) \tag{B2}$$

with Fourier transform

$$X(\omega)=\sigma\sqrt{2\pi}A_0[e^{j\phi_0}e^{-\sigma^2(\omega-\omega_0)^2/2} + e^{-j\phi_0}e^{-\sigma^2(\omega+\omega_0)^2/2}]/2. \quad \text{(B3)}$$

Evaluating (B1) yields

$$\langle\omega\rangle_t=\left[\frac{1}{\sigma}\, e^{-\sigma^2\omega_0^2}(1+\cos(2\phi_0))+\int_{-\sigma\omega_0}^{\sigma\omega_0}\omega_0 e^{-x^2}\, dx\right] \Bigg/ \sqrt{\pi}[1+\cos(2\phi_0)e^{-\sigma^2\omega_0^2}]. \quad \text{(B4)}$$

As $\sigma\rightarrow\infty$, $(1/\sigma)e^{-\sigma^2\omega_0^2}\rightarrow 0$, $e^{-\sigma^2\omega_0^2}\rightarrow 0$, and $\int_{-\sigma\omega_0}^{\sigma\omega_0} e^{-x^2}\, dx\rightarrow\sqrt{\pi}$; therefore $\langle\omega\rangle_t\rightarrow\omega_0$, in agreement with the convergence of the spectral peak of a finite duration tone to ω_0 in the limit.[19]

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