Amplitude and Phase Factorization of Signals via Blaschke Product and Its Applications

Naoki Saito[1] *Jimena Royo Letelier*[2]

[1]Chair of Graduate Group in Applied Mathematics and
Professor of Department of Mathematics
University of California, Davis

[2]Ecole Polytechnique and Ecole Normale Supérieure–Cachan

March 9, 2009

Outline

Outline

Motivation

- Many natural and man-made signals exhibit time-varying frequencies (e.g., chirps, FM radio waves).
- Characterization and analysis of such a signal, $u(t)$, based on instantaneous amplitude $a(t)$, instantaneous phase $\phi(t)$, and instantaneous frequency $\omega(t) := \phi'(t)$ is very important:

$$u(t) = a(t) \cos \phi(t).$$

- The standard discrete wavelet, wavelet packet, and local cosine/sine transforms cannot extract phase information explicitly.
- Want to capture local phase information of sonar signals as well as instantaneous frequency and other features useful for sonar waveform classification.

Motivation

- Many natural and man-made signals exhibit time-varying frequencies (e.g., chirps, FM radio waves).
- Characterization and analysis of such a signal, $u(t)$, based on instantaneous amplitude $a(t)$, instantaneous phase $\phi(t)$, and instantaneous frequency $\omega(t) := \phi'(t)$ is very important:

$$u(t) = a(t) \cos \phi(t).$$

- The standard discrete wavelet, wavelet packet, and local cosine/sine transforms cannot extract phase information explicitly.
- Want to capture local phase information of sonar signals as well as instantaneous frequency and other features useful for sonar waveform classification.

Motivation

- Many natural and man-made signals exhibit time-varying frequencies (e.g., chirps, FM radio waves).
- Characterization and analysis of such a signal, $u(t)$, based on instantaneous amplitude $a(t)$, instantaneous phase $\phi(t)$, and instantaneous frequency $\omega(t) := \phi'(t)$ is very important:

$$u(t) = a(t)\cos\phi(t).$$

- The standard discrete wavelet, wavelet packet, and local cosine/sine transforms cannot extract phase information explicitly.
- Want to capture local phase information of sonar signals as well as instantaneous frequency and other features useful for sonar waveform classification.

Motivation

- Many natural and man-made signals exhibit time-varying frequencies (e.g., chirps, FM radio waves).
- Characterization and analysis of such a signal, $u(t)$, based on instantaneous amplitude $a(t)$, instantaneous phase $\phi(t)$, and instantaneous frequency $\omega(t) := \phi'(t)$ is very important:

$$u(t) = a(t) \cos \phi(t).$$

- The standard discrete wavelet, wavelet packet, and local cosine/sine transforms cannot extract phase information explicitly.
- Want to capture local phase information of sonar signals as well as instantaneous frequency and other features useful for sonar waveform classification.

Outline

Analytic Signal

- It is convenient to use a complexified version of the signal whose real part is a given real-valued signal $u(t)$.
- Given $u(t)$, however, there are infinitely many ways to define the instantaneous amplitude and phase (IAP) pairs so that

$$u(t) = a(t)\cos\phi(t).$$

This is due to the arbitrariness of the complexified version of u, i.e.,

$$f(t) = u(t) + \mathrm{i}v(t)$$

where $v(t)$ is an arbitrary real-valued signal; yet this yields the IAP representation of $u(t)$ via

$$a(t) = \sqrt{u^2(t) + v^2(t)}, \quad \phi(t) = \arctan\frac{v(t)}{u(t)}.$$

The instantaneous frequency is defined as

$$\omega(t) := \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{u(t)v'(t) - u'(t)v(t)}{u^2(t) + v^2(t)}.$$

Analytic Signal

- It is convenient to use a complexified version of the signal whose real part is a given real-valued signal $u(t)$.
- Given $u(t)$, however, there are infinitely many ways to define the instantaneous amplitude and phase (IAP) pairs so that

$$u(t) = a(t)\cos\phi(t).$$

This is due to the arbitrariness of the complexified version of u, i.e.,

$$f(t) = u(t) + \mathrm{i}v(t)$$

where $v(t)$ is an arbitrary real-valued signal; yet this yields the IAP representation of $u(t)$ via

$$a(t) = \sqrt{u^2(t) + v^2(t)}, \quad \phi(t) = \arctan\frac{v(t)}{u(t)}.$$

The instantaneous frequency is defined as

$$\omega(t) := \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{u(t)v'(t) - u'(t)v(t)}{u^2(t) + v^2(t)}.$$

Analytic Signal

- Gabor (1946) proposed to use the the Hilbert transform of $u(t)$ as $v(t)$, and called the complex-valued $f(t)$ an analytic signal.
- Vakman (1972) proved that $v(t)$ must be of the Hilbert transform of $u(t)$ if we impose some a priori physical assumptions:
  1. $v(t)$ must be derived from $u(t)$.
  2. Amplitude continuity: a small change in $u \Longrightarrow$ a small change in $a(t)$.
  3. Phase independence of scale: if $cu(t)$, $c \in \mathbb{R}$ arbitrary scalar, then the phase does not change from that of $u(t)$ and its amplitude becomes c times that of $u(t)$.
  4. Harmonic correspondence: if $u(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) \equiv a_0$, $\phi(t) \equiv \omega_0 t + \phi_0$.

Analytic Signal

- Gabor (1946) proposed to use the the Hilbert transform of $u(t)$ as $v(t)$, and called the complex-valued $f(t)$ an analytic signal.
- Vakman (1972) proved that $v(t)$ must be of the Hilbert transform of $u(t)$ if we impose some a priori physical assumptions:
  1. $v(t)$ must be derived from $u(t)$.
  2. Amplitude continuity: a small change in $u \Longrightarrow$ a small change in $a(t)$.
  3. Phase independence of scale: if $cu(t)$, $c \in \mathbb{R}$ arbitrary scalar, then the phase does not change from that of $u(t)$ and its amplitude becomes c times that of $u(t)$.
  4. Harmonic correspondence: if $u(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) \equiv a_0$, $\phi(t) \equiv \omega_0 t + \phi_0$.

Analytic Signal

- Gabor (1946) proposed to use the the Hilbert transform of $u(t)$ as $v(t)$, and called the complex-valued $f(t)$ an analytic signal.
- Vakman (1972) proved that $v(t)$ must be of the Hilbert transform of $u(t)$ if we impose some a priori physical assumptions:
  1. $v(t)$ must be derived from $u(t)$.
  2. Amplitude continuity: a small change in $u \Longrightarrow$ a small change in $a(t)$.
  3. Phase independence of scale: if $cu(t)$, $c \in \mathbb{R}$ arbitrary scalar, then the phase does not change from that of $u(t)$ and its amplitude becomes c times that of $u(t)$.
  4. Harmonic correspondence: if $u(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) \equiv a_0$, $\phi(t) \equiv \omega_0 t + \phi_0$.

Analytic Signal

- Gabor (1946) proposed to use the the Hilbert transform of $u(t)$ as $v(t)$, and called the complex-valued $f(t)$ an analytic signal.
- Vakman (1972) proved that $v(t)$ must be of the Hilbert transform of $u(t)$ if we impose some a priori physical assumptions:
  1. $v(t)$ must be derived from $u(t)$.
  2. Amplitude continuity: a small change in $u \Longrightarrow$ a small change in $a(t)$.
  3. Phase independence of scale: if $cu(t)$, $c \in \mathbb{R}$ arbitrary scalar, then the phase does not change from that of $u(t)$ and its amplitude becomes c times that of $u(t)$.
  4. Harmonic correspondence: if $u(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) \equiv a_0$, $\phi(t) \equiv \omega_0 t + \phi_0$.

Analytic Signal

- Gabor (1946) proposed to use the the Hilbert transform of $u(t)$ as $v(t)$, and called the complex-valued $f(t)$ an analytic signal.
- Vakman (1972) proved that $v(t)$ must be of the Hilbert transform of $u(t)$ if we impose some a priori physical assumptions:
  1. $v(t)$ must be derived from $u(t)$.
  2. Amplitude continuity: a small change in $u \Longrightarrow$ a small change in $a(t)$.
  3. Phase independence of scale: if $cu(t)$, $c \in \mathbb{R}$ arbitrary scalar, then the phase does not change from that of $u(t)$ and its amplitude becomes c times that of $u(t)$.
  4. Harmonic correspondence: if $u(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) \equiv a_0$, $\phi(t) \equiv \omega_0 t + \phi_0$.

Analytic Signal

- Gabor (1946) proposed to use the the Hilbert transform of $u(t)$ as $v(t)$, and called the complex-valued $f(t)$ an analytic signal.
- Vakman (1972) proved that $v(t)$ must be of the Hilbert transform of $u(t)$ if we impose some a priori physical assumptions:
  1. $v(t)$ must be derived from $u(t)$.
  2. Amplitude continuity: a small change in $u \Longrightarrow$ a small change in $a(t)$.
  3. Phase independence of scale: if $cu(t)$, $c \in \mathbb{R}$ arbitrary scalar, then the phase does not change from that of $u(t)$ and its amplitude becomes c times that of $u(t)$.
  4. Harmonic correspondence: if $u(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) \equiv a_0$, $\phi(t) \equiv \omega_0 t + \phi_0$.

Analytic Signal . . .

- For simplicity, we assume that our signals are 2π-periodic in $\theta \in [-\pi, \pi)$.
- Hence, we work on the unit circle and unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$.
- Note that the signals over $\mathbb{R} = (-\infty, \infty)$ can be treated similarly by considering the real axis and the upper half plane of $\mathbb{C}$.
- The analytic signal of a given signal $u(\theta) \in \mathbb{R}$ is often and simply obtained via the Hilbert transform:

$$f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta), \quad \mathcal{H}u(\theta) := \frac{1}{2\pi}\, \mathrm{pv}\int_{-\pi}^{\pi} u(\tau) \cot \frac{\theta - \tau}{2}\, \mathrm{d}\tau.$$

- Note that

$$u(\theta) = \frac{a_0}{2} + \sum_{k \geq 1} (a_k \cos k\theta + b_k \sin k\theta) \Rightarrow \mathcal{H}u(\theta) = \sum_{k \geq 1} (a_k \sin k\theta - b_k \cos k\theta).$$

- Furthermore,

$$f(\theta) = \frac{a_0}{2} + \sum_{k \geq 1} (a_k - \mathrm{i}b_k) \mathrm{e}^{\mathrm{i}k\theta}.$$

Analytic Signal . . .

- For simplicity, we assume that our signals are 2π-periodic in $\theta \in [-\pi, \pi)$.
- Hence, we work on the unit circle and unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$.
- Note that the signals over $\mathbb{R} = (-\infty, \infty)$ can be treated similarly by considering the real axis and the upper half plane of $\mathbb{C}$.
- The analytic signal of a given signal $u(\theta) \in \mathbb{R}$ is often and simply obtained via the Hilbert transform:

$$f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta), \quad \mathcal{H}u(\theta) := \frac{1}{2\pi}\,\mathrm{pv}\int_{-\pi}^{\pi} u(\tau)\cot\frac{\theta-\tau}{2}\,\mathrm{d}\tau.$$

- Note that

$$u(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k \cos k\theta + b_k \sin k\theta) \Rightarrow \mathcal{H}u(\theta) = \sum_{k\geq 1}(a_k \sin k\theta - b_k \cos k\theta).$$

- Furthermore,

$$f(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k - \mathrm{i}b_k)\mathrm{e}^{\mathrm{i}k\theta}.$$

Analytic Signal . . .

- For simplicity, we assume that our signals are 2π-periodic in $\theta \in [-\pi, \pi)$.
- Hence, we work on the unit circle and unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$.
- Note that the signals over $\mathbb{R} = (-\infty, \infty)$ can be treated similarly by considering the real axis and the upper half plane of $\mathbb{C}$.
- The analytic signal of a given signal $u(\theta) \in \mathbb{R}$ is often and simply obtained via the Hilbert transform:

$$f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta), \quad \mathcal{H}u(\theta) := \frac{1}{2\pi}\,\mathrm{pv}\!\int_{-\pi}^{\pi} u(\tau)\cot\frac{\theta-\tau}{2}\,\mathrm{d}\tau.$$

- Note that

$$u(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k \cos k\theta + b_k \sin k\theta) \Rightarrow \mathcal{H}u(\theta) = \sum_{k\geq 1}(a_k \sin k\theta - b_k \cos k\theta).$$

- Furthermore,

$$f(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k - \mathrm{i}b_k)\mathrm{e}^{\mathrm{i}k\theta}.$$

Analytic Signal . . .

- For simplicity, we assume that our signals are 2π-periodic in $\theta \in [-\pi, \pi)$.
- Hence, we work on the unit circle and unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$.
- Note that the signals over $\mathbb{R} = (-\infty, \infty)$ can be treated similarly by considering the real axis and the upper half plane of $\mathbb{C}$.
- The analytic signal of a given signal $u(\theta) \in \mathbb{R}$ is often and simply obtained via the Hilbert transform:

$$f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta), \quad \mathcal{H}u(\theta) := \frac{1}{2\pi}\,\mathrm{pv}\!\int_{-\pi}^{\pi} u(\tau)\cot\frac{\theta-\tau}{2}\,\mathrm{d}\tau.$$

- Note that

$$u(\theta) = \frac{a_0}{2} + \sum_{k\ge1}(a_k \cos k\theta + b_k \sin k\theta) \Rightarrow \mathcal{H}u(\theta) = \sum_{k\ge1}(a_k \sin k\theta - b_k \cos k\theta).$$

- Furthermore,

$$f(\theta) = \frac{a_0}{2} + \sum_{k\ge1}(a_k - \mathrm{i}b_k)\mathrm{e}^{\mathrm{i}k\theta}.$$

Analytic Signal . . .

- For simplicity, we assume that our signals are 2π-periodic in $\theta \in [-\pi, \pi)$.
- Hence, we work on the unit circle and unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$.
- Note that the signals over $\mathbb{R} = (-\infty, \infty)$ can be treated similarly by considering the real axis and the upper half plane of $\mathbb{C}$.
- The analytic signal of a given signal $u(\theta) \in \mathbb{R}$ is often and simply obtained via the Hilbert transform:

$$f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta), \quad \mathcal{H}u(\theta) := \frac{1}{2\pi}\,\mathrm{pv}\int_{-\pi}^{\pi} u(\tau)\cot\frac{\theta-\tau}{2}\,\mathrm{d}\tau.$$

- Note that

$$u(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k \cos k\theta + b_k \sin k\theta) \Rightarrow \mathcal{H}u(\theta) = \sum_{k\geq 1}(a_k \sin k\theta - b_k \cos k\theta).$$

- Furthermore,

$$f(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k - \mathrm{i}b_k)\mathrm{e}^{\mathrm{i}k\theta}.$$

Analytic Signal . . .

- For simplicity, we assume that our signals are 2π-periodic in $\theta \in [-\pi, \pi)$.
- Hence, we work on the unit circle and unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$.
- Note that the signals over $\mathbb{R} = (-\infty, \infty)$ can be treated similarly by considering the real axis and the upper half plane of $\mathbb{C}$.
- The analytic signal of a given signal $u(\theta) \in \mathbb{R}$ is often and simply obtained via the Hilbert transform:

$$f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta), \quad \mathcal{H}u(\theta) := \frac{1}{2\pi}\, \mathrm{pv}\int_{-\pi}^{\pi} u(\tau) \cot \frac{\theta - \tau}{2}\, \mathrm{d}\tau.$$

- Note that

$$u(\theta) = \frac{a_0}{2} + \sum_{k \geq 1} (a_k \cos k\theta + b_k \sin k\theta) \Rightarrow \mathcal{H}u(\theta) = \sum_{k \geq 1} (a_k \sin k\theta - b_k \cos k\theta).$$

- Furthermore,

$$f(\theta) = \frac{a_0}{2} + \sum_{k \geq 1} (a_k - \mathrm{i}b_k)\mathrm{e}^{\mathrm{i}k\theta}.$$

We can gain a deeper insight by viewing this as the boundary value of an analytic function $F(z)$ where

$$F(z) := U(z) + \mathrm{i}\widetilde{U}(z), \quad z \in \mathbb{D},$$

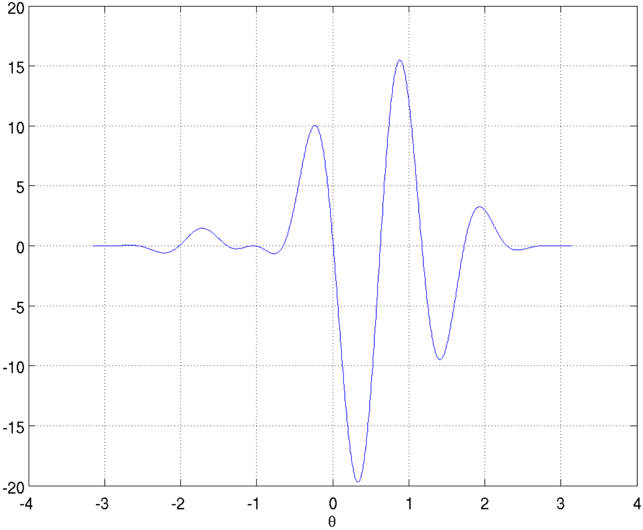
where

$$U(z) = U\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = P_r * u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\tau)+r^2} u(\tau)\,\mathrm{d}\tau,$$

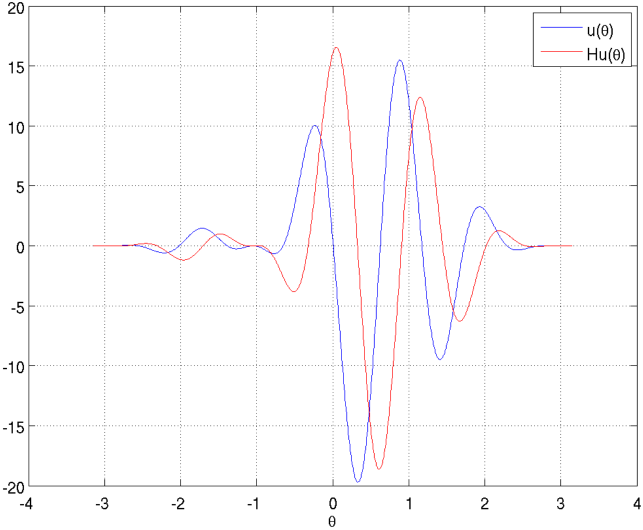
$$\widetilde{U}(z) = \widetilde{U}\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = Q_r * u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r\sin(\theta-\tau)}{1-2r\cos(\theta-\tau)+r^2} u(\tau)\,\mathrm{d}\tau.$$

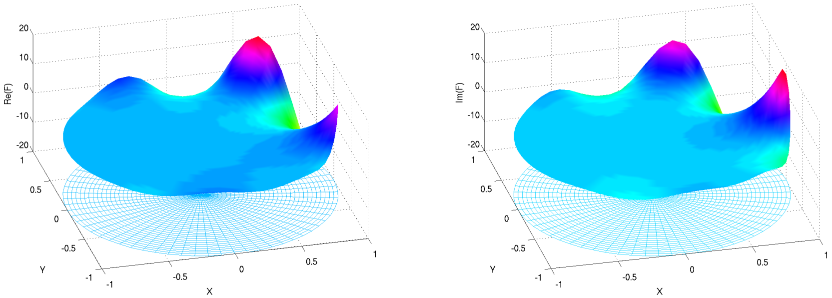
In other words, the original signal $u(\theta) = U\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ is the boundary value of the harmonic function U on $\partial\mathbb{D}$, which is constructed by the Poisson integral. $\widetilde{U}$ and $Q_r(\theta)$ are referred to as the conjugate harmonic function and the conjugate Poisson kernel, respectively.

Analytic Signal . . . An Example: $u(\theta)$



Analytic Signal . . . An Example: $u(\theta)$ and $\mathcal{H}u(\theta)$



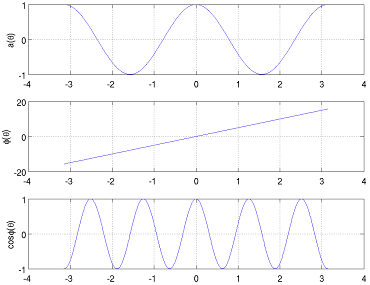


(a) $U(z)$ (b) $\widetilde{U}(z)$

Analytic Signal . . .

Even if we use the analytic signal, its IAP representation is not unique as shown by Cohen, Loughlin, and Vakman (1999):

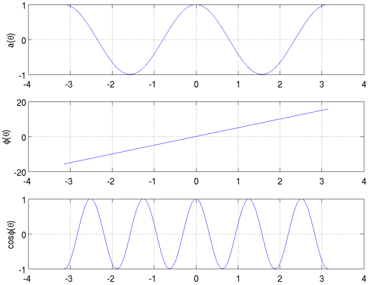
- $f(\theta) = a(\theta)\mathrm{e}^{\mathrm{i}\phi(\theta)}$, where $a(\theta) = u(\theta)\cos\phi(\theta) + v(\theta)\sin\phi(\theta)$ may be negative though $\phi(\theta)$ is continuous;
- $f(\theta) = |a(\theta)|\mathrm{e}^{\mathrm{i}(\phi(\theta)+\pi\alpha(\theta))}$, where $\alpha(\theta)$ is an appropriate phase function, which may be discontinuous.



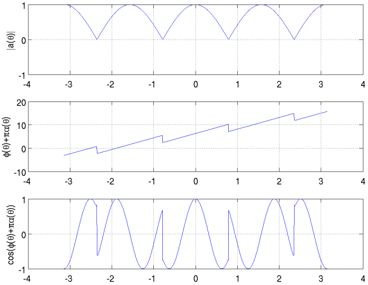
(a) Continuous phase

Even if we use the analytic signal, its IAP representation is not unique as shown by Cohen, Loughlin, and Vakman (1999):

- $f(\theta) = a(\theta)\mathrm{e}^{\mathrm{i}\phi(\theta)}$, where $a(\theta) = u(\theta)\cos\phi(\theta) + v(\theta)\sin\phi(\theta)$ may be negative though $\phi(\theta)$ is continuous;
- $f(\theta) = |a(\theta)|\mathrm{e}^{\mathrm{i}(\phi(\theta)+\pi\alpha(\theta))}$, where $\alpha(\theta)$ is an appropriate phase function, which may be discontinuous.



(a) Continuous phase



(b) Nonnegative amplitude

Outline

Phase Signal (or Blaschke Product)

- Avoiding such ambiguity leads to the concept of phase signal (or the use of the Blaschke product) by Picinbono (1997–8); Kumaresan-Rao (1998–9), Coifman-Nahon (1999–2000).
- Instead of seeking the IAP representation of an analytic signal as $f(\theta) = a(\theta)e^{i\phi(\theta)}$, we seek a more specific form:

$$f(\theta) = b(\theta)g(\theta) \qquad \text{the } \partial\mathbb{D} \text{ version;}$$
$$F(z) = B(z)G(z) \qquad \text{the } \mathbb{D} \text{ version,}$$

where $b(\theta) = B\left(e^{i\theta}\right)$ is called the phase signal and $B(z)$ is called the Blaschke product of $F(z)$.
- The Blaschke product takes care of all the zeros of $F(z)$ in $\mathbb{D}$:

$$B(z) := z^N \cdot \prod_{k=1}^{M} \left(\frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \cdot \frac{\overline{\alpha}_k}{|\alpha_k|} \right),$$

where $\{\alpha_k\}_{k=1}^{M} \subset \mathbb{D}$ are the nonzero roots of $F(z)$. Note that M could be ∞, but in the practical cases, it is finite.

Phase Signal (or Blaschke Product)

- Avoiding such ambiguity leads to the concept of phase signal (or the use of the Blaschke product) by Picinbono (1997–8); Kumaresan-Rao (1998–9), Coifman-Nahon (1999–2000).
- Instead of seeking the IAP representation of an analytic signal as $f(\theta) = a(\theta)\mathrm{e}^{\mathrm{i}\phi(\theta)}$, we seek a more specific form:

$$\begin{aligned} f(\theta) &= b(\theta)g(\theta) && \text{the } \partial\mathbb{D} \text{ version;} \\ F(z) &= B(z)G(z) && \text{the } \mathbb{D} \text{ version,} \end{aligned}$$

  where $b(\theta) = B\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ is called the phase signal and $B(z)$ is called the Blaschke product of $F(z)$.
- The Blaschke product takes care of all the zeros of $F(z)$ in $\mathbb{D}$:

$$B(z) := z^N \cdot \prod_{k=1}^{M} \left(\frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \cdot \frac{\overline{\alpha}_k}{|\alpha_k|} \right),$$

  where $\{\alpha_k\}_{k=1}^{M} \subset \mathbb{D}$ are the nonzero roots of $F(z)$. Note that M could be ∞, but in the practical cases, it is finite.

Phase Signal (or Blaschke Product)

- Avoiding such ambiguity leads to the concept of phase signal (or the use of the Blaschke product) by Picinbono (1997–8); Kumaresan-Rao (1998–9), Coifman-Nahon (1999–2000).
- Instead of seeking the IAP representation of an analytic signal as $f(\theta) = a(\theta)\mathrm{e}^{\mathrm{i}\phi(\theta)}$, we seek a more specific form:

$$\begin{aligned} f(\theta) &= b(\theta)g(\theta) && \text{the } \partial\mathbb{D} \text{ version;} \\ F(z) &= B(z)G(z) && \text{the } \mathbb{D} \text{ version,} \end{aligned}$$

  where $b(\theta) = B\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ is called the phase signal and $B(z)$ is called the Blaschke product of $F(z)$.
- The Blaschke product takes care of all the zeros of $F(z)$ in $\mathbb{D}$:

$$B(z) := z^N \cdot \prod_{k=1}^{M} \left(\frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \cdot \frac{\overline{\alpha}_k}{|\alpha_k|} \right),$$

  where $\{\alpha_k\}_{k=1}^{M} \subset \mathbb{D}$ are the nonzero roots of $F(z)$. Note that M could be ∞, but in the practical cases, it is finite.

Some Properties of the Blaschke Product

- $|b(\theta)| = |B(e^{i\theta})| = 1.$
- In fact, one can show [Coifman-Nahon (2000), Kumaresan-Rao (1999)] that if $B(e^{i\theta}) = e^{i\phi(\theta)}$ for some $\phi : [-\pi, \pi) \to \mathbb{R}$,

$$B(e^{i\theta}) = B(1) \cdot e^{i\int_0^\theta \phi'(t)\,dt}, \quad \phi'(\theta) = N + \sum_{k=1}^{M} \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} > 0,$$

  i.e., the phase $\phi(\theta)$ is non-decreasing, and the instantaneous frequency $\omega(\theta) = \phi'(\theta)$ is nonnegative. Hence, there is no serious phase unwrapping problem.
- $G(z)$ is analytic in $\mathbb{D}$ and contains no zeros there.
- $|g(\theta)| = |G(e^{i\theta})| = |F(e^{i\theta})| = |f(\theta)|.$
- Hence $g(\theta)$ can be viewed as the amplitude of $f(\theta)$, but it is complexed-valued in general.

Some Properties of the Blaschke Product

- $|b(\theta)| = |B(e^{i\theta})| = 1.$
- In fact, one can show [Coifman-Nahon (2000), Kumaresan-Rao (1999)] that if $B(e^{i\theta}) = e^{i\phi(\theta)}$ for some $\phi : [-\pi, \pi) \to \mathbb{R}$,

$$B(e^{i\theta}) = B(1) \cdot e^{i\int_0^\theta \phi'(t)\,dt}, \quad \phi'(\theta) = N + \sum_{k=1}^{M} \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} > 0,$$

i.e., the phase $\phi(\theta)$ is non-decreasing, and the instantaneous frequency $\omega(\theta) = \phi'(\theta)$ is nonnegative. Hence, there is no serious phase unwrapping problem.

- $G(z)$ is analytic in $\mathbb{D}$ and contains no zeros there.
- $|g(\theta)| = |G(e^{i\theta})| = |F(e^{i\theta})| = |f(\theta)|$.
- Hence $g(\theta)$ can be viewed as the amplitude of $f(\theta)$, but it is complexed-valued in general.

Some Properties of the Blaschke Product

- $|b(\theta)| = |B(e^{i\theta})| = 1$.
- In fact, one can show [Coifman-Nahon (2000), Kumaresan-Rao (1999)] that if $B(e^{i\theta}) = e^{i\phi(\theta)}$ for some $\phi : [-\pi, \pi) \to \mathbb{R}$,

$$B(e^{i\theta}) = B(1) \cdot e^{i \int_0^\theta \phi'(t)\, dt}, \quad \phi'(\theta) = N + \sum_{k=1}^{M} \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} > 0,$$

  i.e., the phase $\phi(\theta)$ is non-decreasing, and the instantaneous frequency $\omega(\theta) = \phi'(\theta)$ is nonnegative. Hence, there is no serious phase unwrapping problem.
- $G(z)$ is analytic in $\mathbb{D}$ and contains no zeros there.
- $|g(\theta)| = |G(e^{i\theta})| = |F(e^{i\theta})| = |f(\theta)|$.
- Hence $g(\theta)$ can be viewed as the amplitude of $f(\theta)$, but it is complexed-valued in general.

Some Properties of the Blaschke Product

- $|b(\theta)| = |B(\mathrm{e}^{\mathrm{i}\theta})| = 1$.
- In fact, one can show [Coifman-Nahon (2000), Kumaresan-Rao (1999)] that if $B(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\phi(\theta)}$ for some $\phi : [-\pi, \pi) \to \mathbb{R}$,

$$B(\mathrm{e}^{\mathrm{i}\theta}) = B(1) \cdot \mathrm{e}^{\mathrm{i}\int_0^\theta \phi'(t)\,\mathrm{d}t}, \quad \phi'(\theta) = N + \sum_{k=1}^{M} \frac{1 - |\alpha_k|^2}{|\mathrm{e}^{\mathrm{i}\theta} - \alpha_k|^2} > 0,$$

  i.e., the phase $\phi(\theta)$ is non-decreasing, and the instantaneous frequency $\omega(\theta) = \phi'(\theta)$ is nonnegative. Hence, there is no serious phase unwrapping problem.
- $G(z)$ is analytic in $\mathbb{D}$ and contains no zeros there.
- $|g(\theta)| = |G(\mathrm{e}^{\mathrm{i}\theta})| = |F(\mathrm{e}^{\mathrm{i}\theta})| = |f(\theta)|$.
- Hence $g(\theta)$ can be viewed as the amplitude of $f(\theta)$, but it is complexed-valued in general.

Some Properties of the Blaschke Product

- $|b(\theta)| = |B(\mathrm{e}^{\mathrm{i}\theta})| = 1$.
- In fact, one can show [Coifman-Nahon (2000), Kumaresan-Rao (1999)] that if $B(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\phi(\theta)}$ for some $\phi : [-\pi, \pi) \to \mathbb{R}$,

$$B(\mathrm{e}^{\mathrm{i}\theta}) = B(1) \cdot \mathrm{e}^{\mathrm{i}\int_0^\theta \phi'(t)\,\mathrm{d}t}, \quad \phi'(\theta) = N + \sum_{k=1}^{M} \frac{1 - |\alpha_k|^2}{|\mathrm{e}^{\mathrm{i}\theta} - \alpha_k|^2} > 0,$$

  i.e., the phase $\phi(\theta)$ is non-decreasing, and the instantaneous frequency $\omega(\theta) = \phi'(\theta)$ is nonnegative. Hence, there is no serious phase unwrapping problem.
- $G(z)$ is analytic in $\mathbb{D}$ and contains no zeros there.
- $|g(\theta)| = |G(\mathrm{e}^{\mathrm{i}\theta})| = |F(\mathrm{e}^{\mathrm{i}\theta})| = |f(\theta)|$.
- Hence $g(\theta)$ can be viewed as the amplitude of $f(\theta)$, but it is complexed-valued in general.

An Example

- Let us consider a simple analytic function (in fact a polynomial in z) in $\mathbb{D}$ as

$$F(z) = (z+0.8)^5\,(z-0.98\mathrm{e}^{-\mathrm{i}\pi/3})^2\,(z-0.5\mathrm{e}^{\mathrm{i}\pi/3}).$$

- In this case, we have an explict factorization:

$$B(z) = \left(\frac{z+0.8}{1+0.8z}\right)^5 \left(\frac{z-0.98\mathrm{e}^{-\mathrm{i}\pi/3}}{1-0.98\mathrm{e}^{\mathrm{i}\pi/3}z}\right)^2 \frac{z-0.5\mathrm{e}^{\mathrm{i}\pi/3}}{1-0.5\mathrm{e}^{-\mathrm{i}\pi/3}z};$$

$$G(z) = (1+0.8z)^5(1-0.98\mathrm{e}^{\mathrm{i}\pi/3}z)^2(1-0.5\mathrm{e}^{-\mathrm{i}\pi/3}z).$$

An Example

- Let us consider a simple analytic function (in fact a polynomial in z) in $\mathbb{D}$ as

$$F(z) = (z+0.8)^5\,(z-0.98\mathrm{e}^{-\mathrm{i}\pi/3})^2\,(z-0.5\mathrm{e}^{\mathrm{i}\pi/3}).$$

- In this case, we have an explict factorization:

$$B(z) = \left(\frac{z+0.8}{1+0.8z}\right)^5 \left(\frac{z-0.98\mathrm{e}^{-\mathrm{i}\pi/3}}{1-0.98\mathrm{e}^{\mathrm{i}\pi/3}z}\right)^2 \frac{z-0.5\mathrm{e}^{\mathrm{i}\pi/3}}{1-0.5\mathrm{e}^{-\mathrm{i}\pi/3}z};$$

$$G(z) = (1+0.8z)^5(1-0.98\mathrm{e}^{\mathrm{i}\pi/3}z)^2(1-0.5\mathrm{e}^{-\mathrm{i}\pi/3}z).$$

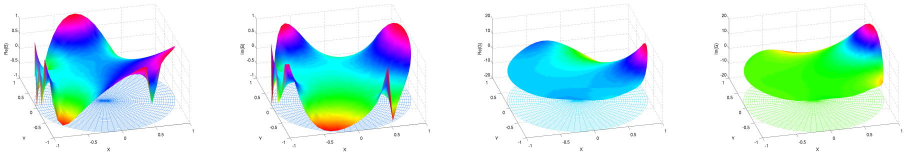
- Let us consider a simple analytic function (in fact a polynomial in z) in $\mathbb{D}$ as

$$F(z) = (z + 0.8)^5\,(z - 0.98\mathrm{e}^{-\mathrm{i}\pi/3})^2\,(z - 0.5\mathrm{e}^{\mathrm{i}\pi/3}).$$

- In this case, we have an explict factorization:

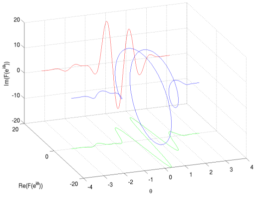
$$B(z) = \left(\frac{z + 0.8}{1 + 0.8z}\right)^5 \left(\frac{z - 0.98\mathrm{e}^{-\mathrm{i}\pi/3}}{1 - 0.98\mathrm{e}^{\mathrm{i}\pi/3}z}\right)^2 \frac{z - 0.5\mathrm{e}^{\mathrm{i}\pi/3}}{1 - 0.5\mathrm{e}^{-\mathrm{i}\pi/3}z};$$

$$G(z) = (1 + 0.8z)^5(1 - 0.98\mathrm{e}^{\mathrm{i}\pi/3}z)^2(1 - 0.5\mathrm{e}^{-\mathrm{i}\pi/3}z).$$

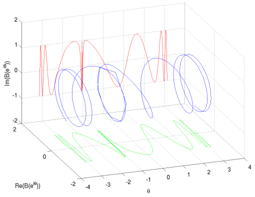


(a) $\mathrm{Re}(B(z))$ (b) $\mathrm{Im}(B(z))$ (c) $\mathrm{Re}(G(z))$ (d) $\mathrm{Im}(G(z))$

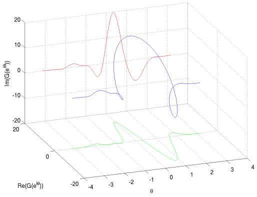
An Example . . .



(a) $F(\mathrm{e}^{\mathrm{i}\theta})$

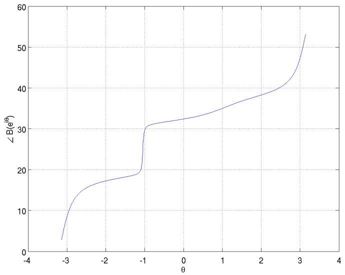


(b) $B(\mathrm{e}^{\mathrm{i}\theta})$

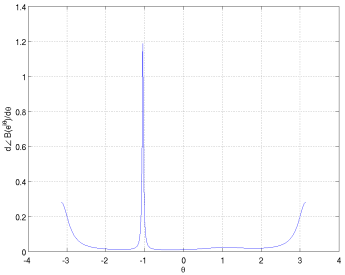


(c) $G(\mathrm{e}^{\mathrm{i}\theta})$

An Example . . . Phase and Instantaneous Frequency



(a) $\phi_b(\theta)$



(b) $\omega_b(\theta)$

Behavior of an Analytic Function $(z - \alpha)^k$ at $\partial\mathbb{D}$

- If $|\alpha| < 1$, it represents k times rotations around the origin of $\mathbb{C}$.
  - If $|\alpha| \ll 1$, then it is close to the pure tones.
  - If $|\alpha| \approx 1$, then amplitude is small around $\theta = \angle\alpha$, and large around $\theta = \angle\alpha \pm \pi$.
- If $|\alpha| > 1$, it does not rotate around the origin, and the lower frequency components are dominant.

Behavior of an Analytic Function $(z - \alpha)^k$ at $\partial\mathbb{D}$

- If $|\alpha| < 1$, it represents k times rotations around the origin of $\mathbb{C}$.
  - If $|\alpha| \ll 1$, then it is close to the pure tones.
  - If $|\alpha| \approx 1$, then amplitude is small around $\theta = \angle\alpha$, and large around $\theta = \angle\alpha \pm \pi$.
- If $|\alpha| > 1$, it does not rotate around the origin, and the lower frequency components are dominant.

Behavior of an Analytic Function $(z-\alpha)^k$ at $\partial\mathbb{D}$

- If $|\alpha| < 1$, it represents k times rotations around the origin of $\mathbb{C}$.
  - If $|\alpha| \ll 1$, then it is close to the pure tones.
  - If $|\alpha| \approx 1$, then amplitude is small around $\theta = \angle\alpha$, and large around $\theta = \angle\alpha \pm \pi$.
- If $|\alpha| > 1$, it does not rotate around the origin, and the lower frequency components are dominant.

Behavior of an Analytic Function $(z-\alpha)^k$ at $\partial\mathbb{D}$

- If $|\alpha| < 1$, it represents k times rotations around the origin of $\mathbb{C}$.
  - If $|\alpha| \ll 1$, then it is close to the pure tones.
  - If $|\alpha| \approx 1$, then amplitude is small around $\theta = \angle\alpha$, and large around $\theta = \angle\alpha \pm \pi$.
- If $|\alpha| > 1$, it does not rotate around the origin, and the lower frequency components are dominant.

The Mathematical Basis of the BG Factorization

Definition (Hardy Spaces)

If $p > 0$, H_p is the set of $F(z)$ analytic in $\mathbb{D}$ with

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta < \infty.$$

Theorem (Herglotz (1911), F. Riesz (1922); see also Hoffman (1962), Koosis (1998), Garnett (2007))

Let $F(z) \not\equiv 0$ belong to H_p, $p > 0$. Then there is a Blaschke product $B(z)$ and a $G(z) \in H_p$ with $F(z) = B(z)G(z)$, where $G(z)$ does not have zeros in $\mathbb{D}$.

Outline

An Algorithm Due to Coifman-Nahon

The original Coifman-Nahon Algorithm

Step 0: For a given real-valued signal $u(\theta)$, compute its analytic signal $f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta)$.

Step 1: Set $\ell(\theta) := \log|f(\theta)|$

Step 2: Compute its analytic version $\ell_a(\theta) := \ell(\theta) + \mathrm{i}\mathcal{H}\ell(\theta)$.

Step 3: Set $g(\theta) := \mathrm{e}^{\ell_a(\theta)}$.

Step 4: Set $b(\theta) := f(\theta)/g(\theta)$.

An Algorithm Due to Coifman-Nahon

The original Coifman-Nahon Algorithm

Step 0: For a given real-valued signal $u(\theta)$, compute its analytic signal $f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta)$.

Step 1: Set $\ell(\theta) := \log|f(\theta)|$

Step 2: Compute its analytic version $\ell_a(\theta) := \ell(\theta) + \mathrm{i}\mathcal{H}\ell(\theta)$.

Step 3: Set $g(\theta) := \mathrm{e}^{\ell_a(\theta)}$.

Step 4: Set $b(\theta) := f(\theta)/g(\theta)$.

The original Coifman-Nahon Algorithm

Step 0: For a given real-valued signal $u(\theta)$, compute its analytic signal $f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta)$.

Step 1: Set $\ell(\theta) := \log|f(\theta)|$

Step 2: Compute its analytic version $\ell_a(\theta) := \ell(\theta) + \mathrm{i}\mathcal{H}\ell(\theta)$.

Step 3: Set $g(\theta) := \mathrm{e}^{\ell_a(\theta)}$.

Step 4: Set $b(\theta) := f(\theta)/g(\theta)$.

An Algorithm Due to Coifman-Nahon

The original Coifman-Nahon Algorithm

Step 0: For a given real-valued signal $u(\theta)$, compute its analytic signal $f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta)$.

Step 1: Set $\ell(\theta) := \log|f(\theta)|$

Step 2: Compute its analytic version $\ell_a(\theta) := \ell(\theta) + \mathrm{i}\mathcal{H}\ell(\theta)$.

Step 3: Set $g(\theta) := \mathrm{e}^{\ell_a(\theta)}$.

Step 4: Set $b(\theta) := f(\theta)/g(\theta)$.

An Algorithm Due to Coifman-Nahon

The original Coifman-Nahon Algorithm

Step 0: For a given real-valued signal $u(\theta)$, compute its analytic signal $f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta)$.

Step 1: Set $\ell(\theta) := \log |f(\theta)|$

Step 2: Compute its analytic version $\ell_a(\theta) := \ell(\theta) + \mathrm{i}\mathcal{H}\ell(\theta)$.

Step 3: Set $g(\theta) := \mathrm{e}^{\ell_a(\theta)}$.

Step 4: Set $b(\theta) := f(\theta)/g(\theta)$.

An Algorithm Due to Coifman-Nahon

The original Coifman-Nahon Algorithm

Step 0: For a given real-valued signal $u(\theta)$, compute its analytic signal $f(\theta) = u(\theta) + \mathrm{i}\mathcal{H}u(\theta)$.

Step 1: Set $\ell(\theta) := \log|f(\theta)|$

Step 2: Compute its analytic version $\ell_a(\theta) := \ell(\theta) + \mathrm{i}\mathcal{H}\ell(\theta)$.

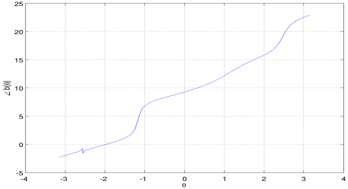
Step 3: Set $g(\theta) := \mathrm{e}^{\ell_a(\theta)}$.

Step 4: Set $b(\theta) := f(\theta)/g(\theta)$.

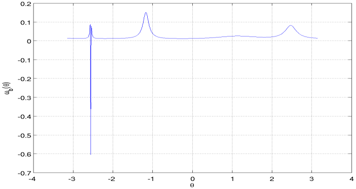
This algorithm can construct $b(\theta)$ modulo multiplicative constants of length 1.

Numerical Instabilities

- If some zeros of $F(z)$ are close to $\partial\mathbb{D}$, numerical instability occurs.
- Coifman and Nahon resolved this by oversampling $f(\theta)$ and $\ell_a(\theta)$, etc., by zero padding in the Fourier domain.

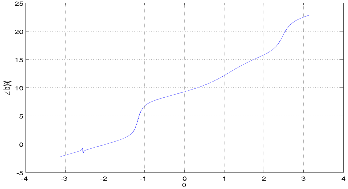


(a) $\phi(\theta)$: No oversampling

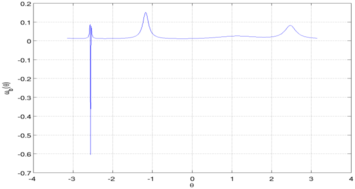


(b) $\omega(\theta)$: No oversampling

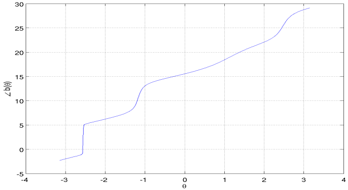
- If some zeros of $F(z)$ are close to $\partial\mathbb{D}$, numerical instability occurs.
- Coifman and Nahon resolved this by oversampling $f(\theta)$ and $\ell_a(\theta)$, etc., by zero padding in the Fourier domain.



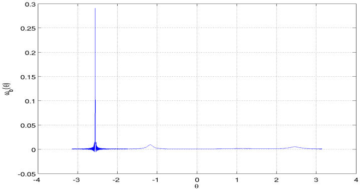
(a) $\phi(\theta)$: No oversampling



(b) $\omega(\theta)$: No oversampling



(c) $\phi(\theta)$: Oversampling



(d) $\omega(\theta)$: Oversampling

Numerical Instabilities . . .

- If $|f(\theta)| \approx 0$ due to inactivity of the original signal or due to the zeros of $F(z)$ too close to $\partial\mathbb{D}$, then $\ell(\theta)$ blows up.
- Coifman and Nahon resolved this by the regularization on $|f(\theta)|$ by

$$|f(\theta)| \leftarrow \sqrt{|f(\theta)|^2 + (\epsilon \|f\|_\infty)^2},$$

where $\epsilon > 0$ is a threshold specified by the user.
- This leads to:

$$|b(\theta)| \begin{cases} \ll 1 & \text{if } |f(\theta)| \ll \epsilon\|f\|_\infty; \\ \approx 1 & \text{if } |f(\theta)| \gg \epsilon\|f\|_\infty \ . \end{cases}$$

Numerical Instabilities . . .

- If $|f(\theta)| \approx 0$ due to inactivity of the original signal or due to the zeros of $F(z)$ too close to $\partial\mathbb{D}$, then $\ell(\theta)$ blows up.
- Coifman and Nahon resolved this by the regularization on $|f(\theta)|$ by

$$|f(\theta)| \leftarrow \sqrt{|f(\theta)|^2 + (\epsilon\|f\|_\infty)^2},$$

where $\epsilon > 0$ is a threshold specified by the user.

- This leads to:

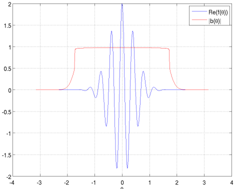
$$|b(\theta)| \begin{cases} \ll 1 & \text{if } |f(\theta)| \ll \epsilon\|f\|_\infty; \\ \approx 1 & \text{if } |f(\theta)| \gg \epsilon\|f\|_\infty \ . \end{cases}$$

- If $|f(\theta)| \approx 0$ due to inactivity of the original signal or due to the zeros of $F(z)$ too close to $\partial\mathbb{D}$, then $\ell(\theta)$ blows up.
- Coifman and Nahon resolved this by the regularization on $|f(\theta)|$ by

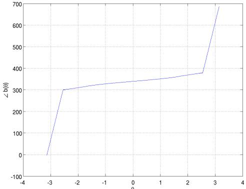
$$|f(\theta)| \leftarrow \sqrt{|f(\theta)|^2 + (\epsilon\|f\|_\infty)^2},$$

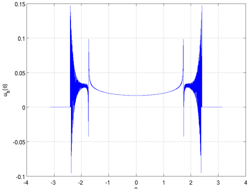
where $\epsilon > 0$ is a threshold specified by the user.
- This leads to:

$$|b(\theta)| \begin{cases} \ll 1 & \text{if } |f(\theta)| \ll \epsilon\|f\|_\infty; \\ \approx 1 & \text{if } |f(\theta)| \gg \epsilon\|f\|_\infty \ . \end{cases}$$



(a) $\epsilon = 0.001$


(b) $\phi(\theta)$


(c) $\omega(\theta)$

Our Contribution

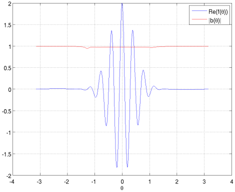
- Selecting appropriate ϵ is not always easy.
- We found that adding a pure sinusoid whose amplitude is small and whose frequency does not interfere too much with the original signal stabilizes numerical algorithm quite well.

Our Contribution

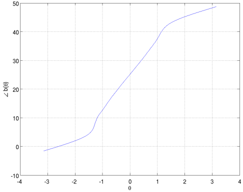
- Selecting appropriate ϵ is not always easy.
- We found that adding a pure sinusoid whose amplitude is small and whose frequency does not interfere too much with the original signal stabilizes numerical algorithm quite well.

Our Contribution

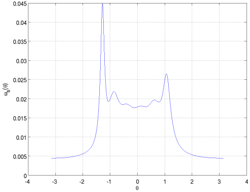
- Selecting appropriate ϵ is not always easy.
- We found that adding a pure sinusoid whose amplitude is small and whose frequency does not interfere too much with the original signal stabilizes numerical algorithm quite well.



(a) Added $0.01\sin(2\theta)$



(b) $\phi(\theta)$



(c) $\omega(\theta)$

Iterative Factorizations

- Let F_0 be the low frequency (or DC) part of F. Then the following factorization is more stable than the simple $F = BG$.

$$F(z) = F_0(z) + B(z) \cdot G(z)$$

- One can iterate the factorization on the G component, i.e.,

$$\begin{aligned} F(z) &= F_0(z) + B_0(z) \cdot G_0(z) \\ &= F_0(z) + B_0(z) \cdot (G_{00}(z) + B_1(z) \cdot G_1(z)) \\ &= F_0(z) + B_0(z) \cdot (G_{00}(z) + B_1(z) \cdot (G_{10}(z) + B_2(z) \cdot G_2(z))) \\ &= \cdots \end{aligned}$$

Iterative Factorizations

- Let F_0 be the low frequency (or DC) part of F. Then the following factorization is more stable than the simple $F = BG$.

$$F(z) = F_0(z) + B(z) \cdot G(z)$$

- One can iterate the factorization on the G component, i.e.,

$$\begin{aligned} F(z) &= F_0(z) + B_0(z) \cdot G_0(z) \\ &= F_0(z) + B_0(z) \cdot (G_{00}(z) + B_1(z) \cdot G_1(z)) \\ &= F_0(z) + B_0(z) \cdot (G_{00}(z) + B_1(z) \cdot (G_{10}(z) + B_2(z) \cdot G_2(z))) \\ &= \cdots \end{aligned}$$

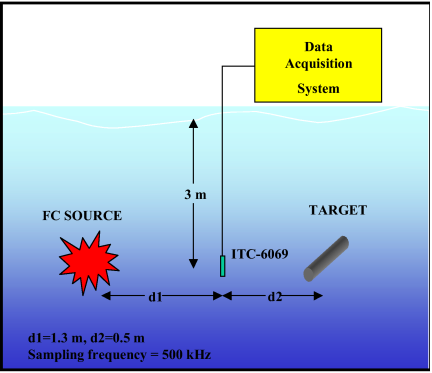
Outline

FC Source Experiment

Purpose: Determine if low cost, "spark-gap" like underwater acoustic source can generate target classification cues.

Approach: Acquire and analyze acoustic backscatter data from multiple dissimilar targets.

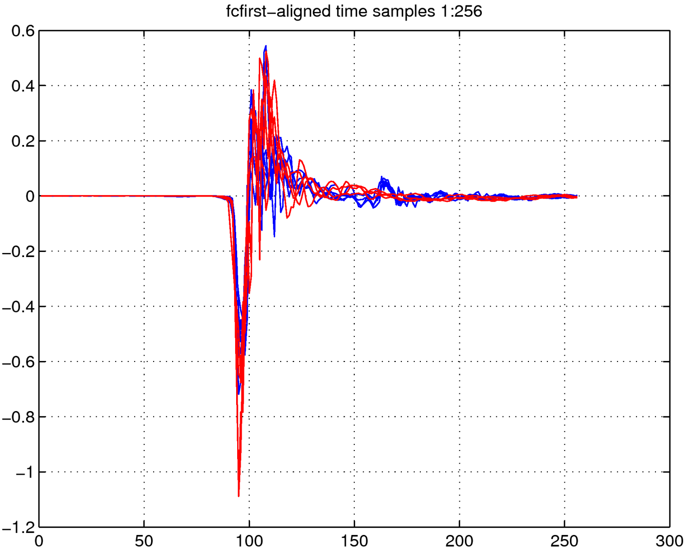
Expected Results: FC source provides enough acoustic energy to generate distinct responses from targets that can provide classification cues.



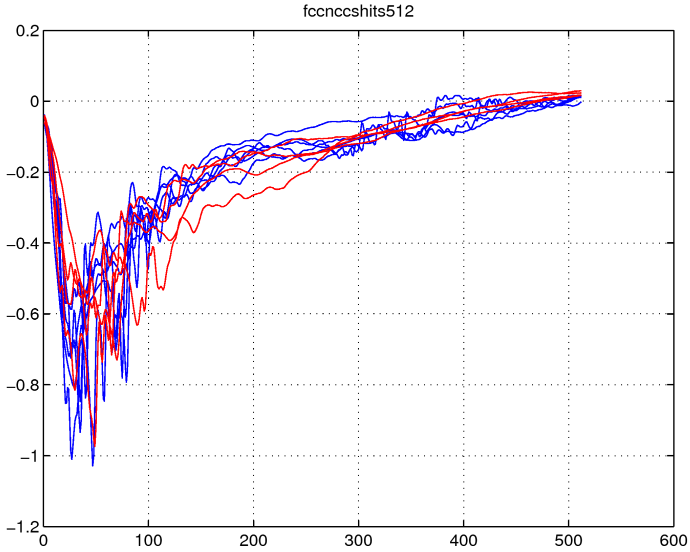
Data Description

- Sampling frequency $= 100$ kHz or 500 kHz, i.e., $\Delta t = 10$ or 2μsec
- There are two sets of data we have been working on: with a target (hollow cylinder containing water) and without a target

Data: 100 kHz Sampling (aligned)



Data: 500 kHz Sampling (aligned)



Our Approach for Classification

- Our observation: reflections from the target may be small and overlap with the reverberation of the direct arrival
- Need to enhance the small reflected waves without amplifying noise
- Our idea: Apply Amplitude-Phase (or *BG*) Factorization via Blaschke Products
- Current Status: Applied the *BG* Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

Our Approach for Classification

- Our observation: reflections from the target may be small and overlap with the reverberation of the direct arrival
- Need to enhance the small reflected waves without amplifying noise
- Our idea: Apply Amplitude-Phase (or *BG*) Factorization via Blaschke Products
- Current Status: Applied the *BG* Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

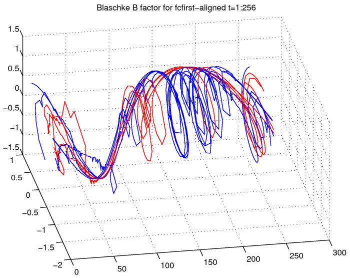
Our Approach for Classification

- Our observation: reflections from the target may be small and overlap with the reverberation of the direct arrival
- Need to enhance the small reflected waves without amplifying noise
- Our idea: Apply Amplitude-Phase (or BG) Factorization via Blaschke Products
- Current Status: Applied the BG Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

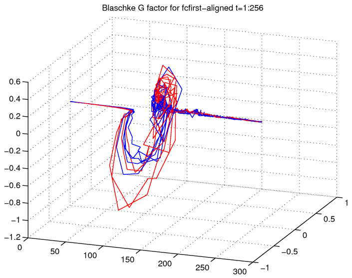
Our Approach for Classification

- Our observation: reflections from the target may be small and overlap with the reverberation of the direct arrival
- Need to enhance the small reflected waves without amplifying noise
- Our idea: Apply Amplitude-Phase (or BG) Factorization via Blaschke Products
- Current Status: Applied the BG Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

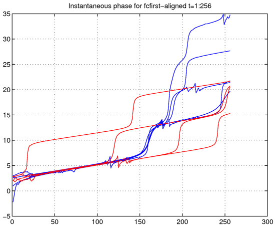
Results of Factorization: 100 kHz Sampling



(a) B_0

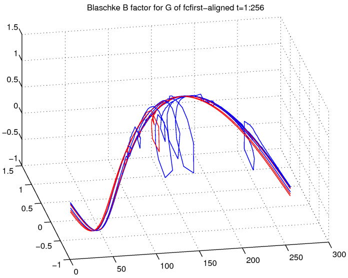


(b) G_0

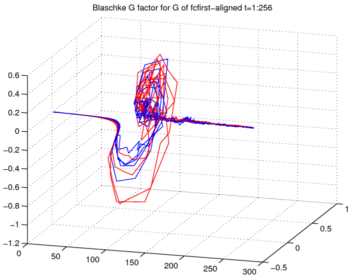


(c) ϕ_0

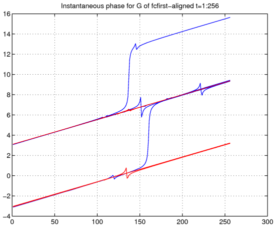
Results of Iterative Factorization: 100 kHz Sampling



(d) B_1

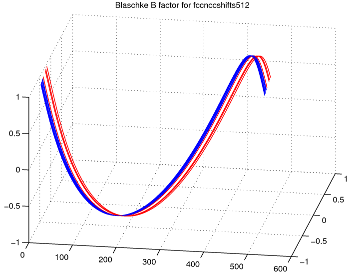


(e) G_1

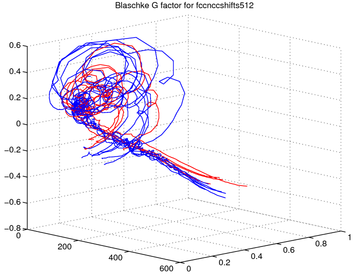


(f) ϕ_1

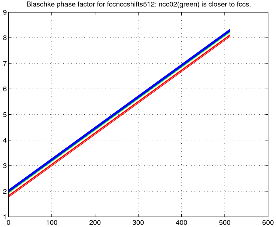
Results of Factorization: 500 kHz Sampling



(a) B_0



(b) G_0



(c) ϕ_0

Outline

Conclusions

- Amplitude-phase factorization via Blaschke product is quite useful for characterization and analysis of time-varying nonstationary signals
- Computed phase monotonically increases in general
- Adding a small pure sinusoid stabilizes the original Coifman-Nahon's algorithm
- Applied the Amplitude-Phase Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

Conclusions

- Amplitude-phase factorization via Blaschke product is quite useful for characterization and analysis of time-varying nonstationary signals
- Computed phase monotonically increases in general
- Adding a small pure sinusoid stabilizes the original Coifman-Nahon's algorithm
- Applied the Amplitude-Phase Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

Conclusions

- Amplitude-phase factorization via Blaschke product is quite useful for characterization and analysis of time-varying nonstationary signals
- Computed phase monotonically increases in general
- Adding a small pure sinusoid stabilizes the original Coifman-Nahon's algorithm
- Applied the Amplitude-Phase Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

Conclusions

- Amplitude-phase factorization via Blaschke product is quite useful for characterization and analysis of time-varying nonstationary signals
- Computed phase monotonically increases in general
- Adding a small pure sinusoid stabilizes the original Coifman-Nahon's algorithm
- Applied the Amplitude-Phase Factorization method successfully to emphasize the small reflected waves or the difference in phase information including time delay

Future Plan

- Investigate the stability of the algorithm against noise
- Investigate the discriminant measure (including Earth Mover's Distance) using phase information for separating out the target/non-target data
- Examine the detailed amplitude-phase diagrams for classification of the materials inside of targets
- Investigate how to localize the phase analysis $\Longrightarrow$ local analytic signals via polyharmonic local Fourier transform
- How about the phase information in 2D/3D signals? $\Longrightarrow$ monogenic signals proposed by Felsberg & Sommer

Future Plan

- Investigate the stability of the algorithm against noise
- Investigate the discriminant measure (including Earth Mover's Distance) using phase information for separating out the target/non-target data
- Examine the detailed amplitude-phase diagrams for classification of the materials inside of targets
- Investigate how to localize the phase analysis $\Longrightarrow$ local analytic signals via polyharmonic local Fourier transform
- How about the phase information in 2D/3D signals? $\Longrightarrow$ monogenic signals proposed by Felsberg & Sommer

Future Plan

- Investigate the stability of the algorithm against noise
- Investigate the discriminant measure (including Earth Mover's Distance) using phase information for separating out the target/non-target data
- Examine the detailed amplitude-phase diagrams for classification of the materials inside of targets
- Investigate how to localize the phase analysis $\Longrightarrow$ local analytic signals via polyharmonic local Fourier transform
- How about the phase information in 2D/3D signals? $\Longrightarrow$ monogenic signals proposed by Felsberg & Sommer

Future Plan

- Investigate the stability of the algorithm against noise
- Investigate the discriminant measure (including Earth Mover's Distance) using phase information for separating out the target/non-target data
- Examine the detailed amplitude-phase diagrams for classification of the materials inside of targets
- Investigate how to localize the phase analysis $\Longrightarrow$ local analytic signals via polyharmonic local Fourier transform
- How about the phase information in 2D/3D signals? $\Longrightarrow$ monogenic signals proposed by Felsberg & Sommer

Future Plan

- Investigate the stability of the algorithm against noise
- Investigate the discriminant measure (including Earth Mover's Distance) using phase information for separating out the target/non-target data
- Examine the detailed amplitude-phase diagrams for classification of the materials inside of targets
- Investigate how to localize the phase analysis $\Longrightarrow$ local analytic signals via polyharmonic local Fourier transform
- How about the phase information in 2D/3D signals? $\Longrightarrow$ monogenic signals proposed by Felsberg & Sommer

Outline

Acknowledgment

- Raphy Coifman (Yale), Michel Nahon (formerly Yale)
- US Navy: Quyen Huynh, Richard Holtzapple
- Support from National Science Foundation and Office of Naval Research

Thank you very much for your attention!