Algebraic approach: starting point is a $*$-algebra $\mathcal{A}$(unital associative algebra with involution $*$)
Cone of states $\mathcal{C}$- linear functionals $\omega \in \mathcal{L}$ obeying $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. (Here $\mathcal{L} = \mathcal{A}^\vee$-dual space.)
Set of normalized states $\mathcal{N}$: normalization condition $\omega(1) = 1$.
Geometric approach: starting point is a convex set of states $\mathcal{N}$ or cone of states $\mathcal{C}$ that are subsets of Banach space $\mathcal{L}$
Evolution operators T_τ-automorphisms
Decoherence from interactions with adiabatic random perturbations. Derivation of probabilities from decoherence.

Classical theories with restricted set of observables (our devices allow us to measure only a part of observables).
Quantum mechanics and its generalizations from such theories.
Geometric theories with commutative group of time translations and spatial translations $\rightarrow$ QFT.
Particles as elementary excitations of ground state.
Quasiparticle as elementary excitations of translation-invariant stationary state.
Inclusive scattering matrix.

Inclusive cross section= probability density of the process
$(M, N) \rightarrow (P, Q, ..., R)$+something
Can be obtained as a limit of matrix elements of inclusive scattering matrix
Inclusive scattering matrix can be expressed in terms of generalized Green functions on shell (an analog of LSZ formula)
Rediscovered in:
What can be measured asymptotically?
S Caron-Huot, M Giroux, HS Hannesdottir, S Mizera
Journal of High Energy Physics 2024 (1), 1-63
Asymptotic observables ≈ inclusive scattering matrix

Green function = expectation value of chronological product (times descending),

$G_n(\mathbf{x}_1, ..., \mathbf{x}_n) = \omega\big(T(A_1(\mathbf{x}_1, t_1)...A_n(\mathbf{x}_n, t_n)\big)$

$A_i \in \mathcal{A}$, ω-translation-invariant stationary state

Generalized Green functions $\omega(MN)$

M-chronological product, N-antichronological product (times ascending).

Appear in Keldysh formalism of non-equilibrium statistical physics and in formalism of L-functionals

If the theory has particle interpretation the inclusive scattering matrix carries the same information as the conventional scattering matrix.
However, inclusive scattering matrix can exist even if the conventional scattering matrix does not exist (for example, for quasiparticles).
In QED conventional scattering matrix does not exist (every process involving fixed number of particles has zero probability).
However, inclusive scattering matrix does exist.

L-functionals

Let us quantize a classical theory with finite or infinite number of degrees of freedom.

If p_k, q^k have standard Poisson brackets after quantization, we obtain operators $\hat{p}_k, \hat{q}^k$ obeying canonical commutation relations (CCR).

We are working with operators $\hat{a}(f) = \int dkf(k)\hat{a}(k), \hat{a}^+(f) = \int dkf(k)\hat{a}^+(k)$ where f runs over the space of test functions $\mathcal{E}$ considered as pre-Hilbert space. The integral over k is considered as an integral over continuous parameters and a sum over discrete parameters.

The CCR can be written in the form $[\hat{a}(f), \hat{a}(g)] = [\hat{a}^+(f), \hat{a}^+(g)] = 0, [\hat{a}(f), \hat{a}^+(\bar{g})] = \hbar\langle f, g\rangle$ where $f, g \in \mathcal{E}$.

In the case of an infinite number of degrees of freedom, there exist representations of CCR that are not equivalent to the standard Fock representation where $\hat{a}(f), \hat{a}^+(f)$ can be interpreted as annihilation and creation operators (i.e. there exists a cyclic vector θ obeying $\hat{a}(f)\theta = 0$)). Vectors and density matrices in all representation spaces can be regarded as states of the theory at hand.

We can represent the states as functionals

$$\mathbf{L}_K(f) = Tr\hat{W}_f K.$$

where $\hat{W}_f = e^{-\hat{a}^+(f)}e^{\hat{a}(\bar{f})}$. It is easy to verify that this functional is well-defined for a density matrix K in any representation of CCR.
One can say that when working with functionals $\mathbf{L}$ we consider all representations of CCR simultaneously.
To emphasize that $\mathbf{L}_K$ does not depend analytically on f we use the notation $\mathbf{L}_K(\bar{f}, f)$ or $\mathbf{L}_K(f^*, f)$.

Weyl algebra=algebra generated by $a(f), a^+(f)$ obeying CCR.
Exponential form of Weyl algebra = algebra $\mathcal{W}$ of operators in Fock space containing all operators of the form W_f and closed in norm topology.
$\mathcal{N}$ is the set of normalized positive linear functionals σ on $\mathcal{W}$ represented by non-linear functionals $\sigma(W_f)$ on $\mathcal{E}$.
The space $\mathcal{L}$ should be identified with the space of linear functionals on $\mathcal{W}$ or with the space of non-linear functionals on $\mathcal{E}$.

$$\frac{d\mathbf{L}}{dt} = H\mathbf{L}$$

H-"Hamiltonian". It can be expressed in terms of operators c_i^+, c_i where $c_1^+(\bar{f})$ is a multiplication operator by $\bar{f}$, $c_2^+(f)$ is a multiplication operator by f, and $c_1(\bar{f}), c_2(f)$ are variational derivatives with respect to $\bar{f}$ and f.

Example: QED when we neglect the action of photons on electrons (joint work with I.Frolov).
Time-dependent Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V} = \hbar \int dk \epsilon(\mathbf{k}) a^+(\mathbf{k}) \cdot a(\mathbf{k})$$
$$+\hbar \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}} (j(\mathbf{k},t) \cdot a^+(\mathbf{k}) + j^*(\mathbf{k},t) \cdot a(\mathbf{k}))$$

where $a(\mathbf{k})$ is a vector potential of electromagnetic field with components $a_\mu(\mathbf{k}), \mu = 0, .., 3$ satisfying the Lorenz gauge condition $k^\mu a_\mu(\mathbf{k}) = 0$. The scalar product of two 4-vectors has the form $p \cdot k = \mathbf{pk} - p_0 k_0$. We use the notation $\epsilon(\mathbf{k}) = |\mathbf{k}|$. We suppose that $j^\mu(\mathbf{k}, t)$ is a numerical function (Fourier transform of divergence-free current.)

The equation of motion for L-functional corresponding to this Hamiltonian has the form , $d\mathbf{L}/dt = H\mathbf{L}$ where $H = H_0 + V$ and

$$H_0 = \hbar \int d\mathbf{k}\epsilon(\mathbf{k})(c_1^+(\mathbf{k}) \cdot c_1(\mathbf{k}) - c_2^+(\mathbf{k}) \cdot c_2(\mathbf{k})),$$

$$V = \hbar \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}}(j(\mathbf{k}, t) \cdot c_1^+(\mathbf{k}) + j^*(\mathbf{k}, t) \cdot c_2^+(\mathbf{k}))$$

In the interaction picture we obtain the following equation for the evolution operator $S(t)$
$i\frac{dS}{dt} = VS(t)$ where
$V = \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}}(e^{i\epsilon(\mathbf{k})t}j(\mathbf{k}, t) \cdot c_1^+(\mathbf{k}) + e^{-i\epsilon(\mathbf{k})t}j^*(\mathbf{k}, t) \cdot c_2^+(\mathbf{k}))$

A solution of this equation can be found in the form $e^{\sum_{i=1}^{2}(M_{i+}(t)\cdot c_i^{+}+M_{i-}(t)\cdot c_i)}$

We obtain

$$L(\alpha^*, \alpha, t) = \exp\Big(\int_{t_0}^{t} d\tau \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}}\big(e^{i\epsilon(\mathbf{k})\tau} j(\mathbf{k}, \tau)\cdot \alpha^*(\mathbf{k}) + e^{-i\epsilon(\mathbf{k})\tau} j^*(\mathbf{k}, \tau)\cdot \alpha(\mathbf{k})\big)\Big) L(\alpha^*, \alpha, t_0)$$

The formula for the solution can be rewritten in the form

$L(\alpha^*, \alpha, t) =$
$\exp\big(\int d\mathbf{k}\sqrt{2\epsilon(\mathbf{k})}(e^{-i\epsilon(\mathbf{k})t}\mathcal{A}(\mathbf{k}, t)\dot{\alpha}^*(\mathbf{k}) +$
$e^{i\epsilon(\mathbf{k})t}\mathcal{A}^*(\mathbf{k}, t)\cdot\alpha(\mathbf{k})\big)L(\alpha^*, \alpha, t_0)$

We use the notation

$$\mathcal{A}^\mu(\mathbf{k}, t) = \frac{1}{2\epsilon(\mathbf{k})(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t} d\tau(e^{i\epsilon(\mathbf{k})(\tau - t)}j^\mu(\mathbf{k}, \tau).$$

where $\mathcal{A}^\mu(\mathbf{k}, t)$ is the expectation value of electromagnetic potential.

We obtain the inclusive cross-section

$$dN(\mathbf{k}) = \mathcal{A}(\mathbf{k}, t) \cdot \mathcal{A}^*(\mathbf{k}, t) 2\epsilon(\mathbf{k}) d\mathbf{k}$$

We calculated the expectation value of the operator

$$\rho(\mathbf{k}) = \sum_{i=\pm} (\varepsilon_i^* \cdot a^+(\mathbf{k}))(\varepsilon_i \cdot a(\mathbf{k})),$$

where ε_i are polarizations of outgoing photons.

If we are interested in the inclusive cross-section of emission of n photons with momenta $k_1, ..., k_n$ then similar calculations lead to the following formula:

$$dN(\mathbf{k}_1, ..., \mathbf{k}_n) = \prod_{i=1}^{n} \mathcal{A}(\mathbf{k}_i, t) \cdot \mathcal{A}^*(\mathbf{k}_i, t) 2\epsilon(\mathbf{k}_i) d\mathbf{k}.$$

QED

$$S = S_{mat} + S_{ph} + \int dx j^{\mu}(x) A_{\mu}(x)$$

In formalism of L-functionals we have doubling of fields.
Inclusive scattering matrix is finite. It can be expressed in terms of generalized Green functions,

Adiabatic scattering matrix for the Hamiltonian $\hat{H}_0 + \hat{V}$ =evolution operator in interaction picture for the time-dependent Hamiltonian

$\hat{H}(t) = \hat{H}_0 + h(at)\hat{V}$

When $a \to 0$ then $h(at)$ changes adiabatically (slowly)

Adiabatic scattering matrix in the formalism of L-functionals= evolution operator in interaction picture for the "Hamiltonian" $H_0 + h(at)V$.

Scattering matrix and inclusive scattering matrix are limits of adiabatic scattering matrices multiplied by some simple factors (Likhachev, Tyupkin, Sch)

Scattering matrix $\hat{S}$ can be expressed in terms of the adiabatic scattering matrix in finite volume Ω in the following way

$$\hat{S} = \lim_{a \to 0} \lim_{\Omega \to \infty} \frac{\hat{U}_{a,\Omega} \hat{S}_{a,\Omega} \hat{U}_{a,\Omega}}{\langle \theta | \hat{S}_{a,\Omega} | \theta \rangle}$$

where

$$\hat{U}_{a,\Omega} = e^{i \sum_k r_{a,\Omega}(k) a^+(k) a(k)},$$

the limit is understood as the convergence of matrix elements in the sense of generalized functions.

Inclusive scattering matrix S *can be represented in the form*

$$S = \lim U_a S_a U_a$$

where

$$U_a = e^{i \int dp r_a(p)(c_1^+(p)c_1(p) - c_2^+(p)c_2(p))}$$

and the function $r_a(p)$ *is chosen in such a way that one-particle L-functionals are* S*-invariant*. Namely, one can take

$$r_a(p) = \int_{-\infty}^{0} d\tau(\epsilon(p, h(a\tau)) - \epsilon(p))$$

where $\epsilon(p, g)$ are one-particle energies of the Hamiltonian $\hat{H}(g) = \hat{H}(0) + g\hat{V}$

Relation between inclusive scattering matrix S and conventional scattering matrix $\hat{S}$

$$SL_K = L_{\hat{S}^* K \hat{S}}$$