How to find coordinate vectors

Example: Suppose we take the standard basis for M_2 given by:

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

and then a second basis given by:

$$T = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\}.$$

Now, if I take an arbitrary vector in M_2,

$$v = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then its coordinate vector with respect to S is easy to compute, $(v)_S = (a, b, c, d)$.

If I want to find $(v)_T$, I use the following algorithm.

1. Write the coordinate vectors of the elements of T in the standard basis:

$$\{(1, 1, 0, 0), (0, 2, -2, 0), (3, 0, -1, 0), (1, 2, 3, 4)\}$$

2. Write these vectors as the columns of a 4×4 matrix adjoined by the vector (a, b, c, d) as a column:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & a \\ 1 & 2 & 0 & 2 & b \\ 0 & -2 & -1 & 3 & c \\ 0 & 0 & 0 & 4 & d \end{array} \right).$$

3. Perform elementary row operations to transform the matrix on the left into the identity. In this example, we obtain:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{1}{2}(a - 3b + 3c - d) \\ 0 & 1 & 0 & 0 & \frac{1}{4}(a - b + 3c - 2d) \\ 0 & 0 & 1 & 0 & \frac{1}{4}(2a - 2b + 2c - d) \\ 0 & 0 & 0 & 1 & \frac{1}{4}d \end{array} \right).$$

4. The adjoining vector at the end of the process is $(v)_T$. So, in this example,

$$(v)_T = \left(-\frac{1}{2}(a - 3b + 3c - d), \frac{1}{4}(a - b + 3c - 2d), \frac{1}{4}(2a - 2b + 2c - d), \frac{1}{4}d \right).$$

The change-of-basis matrix from the basis S to the basis T is then given by:

$$\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$