Final solutions

1. **Short answers.** Show any work.

   (a) (1 pt) Explain why $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2\}$ is not a function.
   Solution: R fails both requirements for being a function. First of all, the domain is not all of $\mathbb{R}$ (only positive x are in the domain). Second of all, one x can map to multiple y's. For example, $(4, 2) \in R$ and $(4, -2) \in R$.

   (b) Let $f = \{(1, 1), (2, 3), (3, 7), (4, 3)\}$.

      i. (1 pt) Find $f(f^{-1}(\{2, 3\}))$.
      Solution: $f^{-1}(\{2, 3\}) = \{2, 4\}$, so $f(f^{-1}(\{2, 3\})) = \{3\}$.

      ii. (1 pt) Find $f(\{1, 2\}) \setminus f(\{4\})$.
      Solution: $f(\{1, 2\}) = \{1, 3\}$, and $f(\{4\}) = \{3\}$, so $f(\{1, 2\}) \setminus f(\{4\}) = \{1\}$.

   (c) i. (1 pt) Give an example of a set A so that $|\mathbb{R}| < |A|$.
      Solution: One answer is $A = \mathcal{P}(\mathbb{R})$. We showed that power sets always have larger cardinality than the original set. Another answer is $V = \{f \subseteq \mathbb{R} \times \mathbb{R} \mid f \text{ is a function.}\}$.

      ii. (2 pts) Give an example of sets $B \subseteq A$ with $|A| = |B|$, but $B \neq A$.
      Solution: First note that both sets A and B have to be infinite. One example is to choose $A = \mathbb{N}$ and B to be the set of all even (or odd) natural numbers. Then $B \subseteq A$, $B \neq A$, but $|A| = |B| = \aleph_0$.
      Another example is to choose $A = \mathbb{R}$ and $B = (0, 1)$. We proved in the homework that there is a bijection between these sets so they have the same cardinality.

(d) Let $A = \{a, b, c\}$.

i. (3 pts) Next to the following sets, write their cardinalities.

$$\begin{array}{rl} 3 & A \\ 2^3 = 8 & \mathcal{P}(A) \\ 3 \times 3 = 9 & A \times A \\ 3^3 = 27 & \text{the set } A^A \text{ consisting of all functions on } A \\ 2^{3\times 3} = 2^9 & \text{the set } \mathcal{R} \text{ consisting of all relations on } A \\ 5 & \text{the set } \mathcal{E} \text{ consisting of all equivalence relations on } A. \end{array}$$

ii. (4 pts) List all the partitions of the set A, and all the equivalence relations on A. Identify which partitions correspond to which equivalence relations.

Solution:

Partitions	$\leftrightarrow$	Equivalence relations
$\{\{1,2,3\}\}$	$\leftrightarrow$	$\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$
$\{\{1,2\},\{3\}\}$	$\leftrightarrow$	$\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$
$\{\{1,3\},\{2\}\}$	$\leftrightarrow$	$\{(1,1),(2,2),(3,3),(1,3),(3,1)\}$
$\{\{2,3\},\{1\}\}$	$\leftrightarrow$	$\{(1,1),(2,2),(3,3),(2,3),(3,2)\}$
$\{\{1\},\{2\},\{3\}\}$	$\leftrightarrow$	$\{(1,1),(2,2),(3,3)\}$

iii. (4 pts) Define the equivalence relation $R \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ by $(X, Y) \in R$ iff there exists a bijection $f : X \to Y$. How many equivalence classes are there in $\mathcal{P}(A)/R$? List them.

Solution: This equivalence relations partitions the subsets of A by cardinalities. Therefore, there is one equivalence class for each possible cardinality. So, there are 4 equivalence classes, one for each of the cardinalities 0,1,2,3.

$$\begin{array}{rl} 0: & \{\emptyset\} \\ 1: & \{\{1\},\{2\},\{3\}\} \\ 2: & \{\{1,2\},\{1,3\},\{2,3\}\} \\ 3: & \{\{1,2,3\}\} \end{array}$$

2. **Definitions.** Complete the following definitions. Be precise.

   (a) (2 pts) The relation $R \subseteq A \times A$ is *transitive*:
   Solution: $\forall x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

   (b) (2 pts) The relation $f \subseteq A \times B$ is a *function*:
   Solution: the domain of f equals all of A, and $\forall x \in A$, $\forall y, z \in B$, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

   (c) (2 pts) The function $f : A \to B$ is a *surjection* (onto):
   Solution: $\forall b \in B$, $\exists a \in A$, so that $(a, b) \in f$ (or $f(a) = b$).

   (d) (2 pts) The function $f : A \to B$ is *invertible*:
   Solution: the inverse relation $f^{-1} \subseteq B \times A$ is actually a function.

(e) (2 pts) If $f : A \to B$ is a function, and $Y \subseteq B$, then $f^{-1}(Y)$ is:
Solution: $\{x \in A \mid f(x) \in Y\}$.

(f) (2 pts) $\mathcal{P} = \{p_\alpha \mid \alpha \in \Omega\}$ is a *partition* of the set A:
Solution: for each α, $p_\alpha \neq \emptyset$, for all $\alpha \neq \beta$, $p_\alpha \cap p_\beta = \emptyset$, and $\cup_{\alpha \in \Omega} p_\alpha = A$.

(g) (2 pts) The set A is *countably infinite*:
Solution: there exists a bijection $f : \mathbb{N} \to A$.

(h) (2 pts) Given the sets A and B, $|A| \leq |B|$:
Solution: there exists an injection $f : A \to B$.

3. **Proofs.** Prove the following theorems. Explain your reasoning.

(a) (5 pts) Let R be an equivalence relation on the set A. Write down the definition of an arbitrary equivalence class x/R. Next, prove if $(x, y) \in R$ then $x/R = y/R$.

Solution: $x/R = \{y \in A \mid (x, y) \in R\}$.

Let $z \in x/R$, which means $(x, z) \in R$. Then, since $(x, y) \in R$ by assumption and R is symmetric, we have $(y, x) \in R$. $(x, z) \in R$ and $(y, x) \in R$ implies $(y, z) \in R$ by transitivity of R. Therefore, $z \in y/R$. We have proved $x/R \subseteq y/R$.

Now let $z \in y/R$. Thus $(y, z) \in R$. Since $(x, y) \in R$ by assumption, and since R is transitive, this means $(x, z) \in R$. Therefore, $z \in x/R$. We have proved $y/R \subseteq x/R$.

Since $x/R \subseteq y/R$ and $y/R \subseteq x/R$, $x/R = y/R$.

(b) (5 pts) Let $f : A \to B$ be a function, with $E, F \subseteq B$. Prove:

$$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F).$$

Solution:

$$\begin{aligned} x \in f^{-1}(E \cup F) &\Leftrightarrow f(x) \in E \cup F \\ &\Leftrightarrow f(x) \in E \text{ or } f(x) \in F \\ &\Leftrightarrow x \in f^{-1}(E) \text{ or } x \in f^{-1}(F) \\ &\Leftrightarrow x \in f^{-1}(E) \cup f^{-1}(F). \end{aligned}$$

(c) (4 pts) Prove the set of odd natural numbers is countably infinite.

Solution: Denote the set of odd natural numbers by the letter $O = \{1, 3, 5, 7, \dots\}$. Define the bijection $f : \mathbb{N} \to O$ by $f(n) = 2n - 1$. (Note: I didn't use $2n + 1$ because then 1 would not be in the range.) This map is surjective, because every odd number has a decomposition into the form $2n - 1$ for some $n \in \mathbb{N}$, and the map is injective because $2n - 1 = 2m - 1$ implies $n = m$.

(d) (5 pts) Let A be a set. Prove $|A| \leq |\mathcal{P}(A)|$. (This is much easier than proving $|A| < |\mathcal{P}(A)|$; don't do more work than you need to.)

Solution: We need to define an injection $f : A \to \mathcal{P}(A)$. One way to do this is to map each element into the singleton set just containing that element, i.e. $f(a) = \{a\}$. This map is injective because $\{a\} = \{b\}$ implies $a = b$.

4. **Pigeonhole principle.**

(a) (3 pts) State the pigeonhole principle.

Solution: Let A be a set of cardinality n, and B be a set of cardinality m, where both n and m are natural numbers with $n > m$. Let $f : A \to B$ be any function. Then f cannot be injective.

(b) (5 pts) Carefully use the pigeonhole principle to prove the following theorem: if any five integers are chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, two of them must add up to 9.

Solution: Let A be the set consisting of the five integers from $\{1, 2, 3, 4, 5, 6, 7, 8\}$, and let B the the set consisting of the four elements $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, and $\{4, 5\}$. Define the function $f : A \to B$ which takes an integer n and places it into the set in B which contains n. This map is well-defined because the sets in B are disjoint (so one integer can't map to more than one element of B). Since $|A| > |B|$ and both are finite, the pigeonhole principle states that the f can't be injective. Therefore, at least two integers map to the same set in B. Since at most two elements in A can map to one element of B, we have exactly two integers mapping to the same set in B. Since the integers in each set of B add up to 9, these two numbers must add up to 9.