Final Solutions

1. **Calculations.** Find the following quantities. No proof required, but show any sidework for partial credit.

   (a) (3 pts) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n}$.

   Solution:

   $$\sum_{n=2}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - 1 - \frac{1}{2} = \frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2} = \frac{1}{2}.$$

   (b) (4 pts) The limit of the sequence $\left(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right)$.

   Solution: The sequence can be written recursively as $a_{n+1} = \sqrt{2a_n}$. So, assuming (a_n) converges, letting $a = \lim a_n$, we get the equation $a = \sqrt{2a}$. The solutions are $a = 0$ or $a = 2$. Because the sequence is increasing and starts out positive, the limit must be $a = 2$.

   (c) (3 pts) The value $\limsup s_n$ where $s_n = (-1)^n \frac{4n-3}{3n+4}$.

   Solution: The limit of $\frac{4n-3}{3n+4}$ is $\frac{4}{3}$, so the sequence (s_n) eventually oscillates between values close to $\frac{4}{3}$ and values close to $-\frac{4}{3}$. Therefore, $\limsup s_n = \frac{4}{3}$.

   (d) (2 pts) The set of limit points of $S = [0,1) \cup \{3\}$.

   The set of limit points is $[0,1]$. Note that 3 is not a limit point, it's an isolated point.

   (e) (3 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

   This can be written as a telescoping series whose limit is 1. (see homework 5 solutions.)

2. Suppose (a_n) is a sequence of real numbers so that $\limsup a_n = \alpha$. Assume $\alpha \in \mathbb{R}$.

   (a) (5 pts) Let $\epsilon > 0$ be arbitrary. Prove by contradiction that there exist infinitely many sequence elements in the neighborhood $(\alpha - \epsilon, \alpha + \epsilon)$.

   Solution: Suppose $\exists \epsilon > 0$ so that there are only finitely many n so that $a_n \in (\alpha - \epsilon, \alpha + \epsilon)$. Let's label then $a_{n_1}, \dots a_{n_k}$. Then for $n > \max\{n_1, \dots n_k\}$, a_n is not in $(\alpha - \epsilon, \alpha + \epsilon)$. Therefore,

$$\sup \{a_n : n > \max\{n_1, \dots n_k\}\}$$

   is either less than or equal to $\alpha - \epsilon$ or greater than or equal to $\alpha + \epsilon$. Therefore, $\limsup a_n$ is either less than or equal to $\alpha - \epsilon$ or greater than or equal to $\alpha + \epsilon$. This is a contradiction to $\limsup a_n = \alpha$.

   (b) (5 pts) Construct a subsequence of (a_n) which converges to α. (Your subsequence does not need to be monotone.)

   Solution: Let $a_{n_0} = a_0$. Inductively, choose $a_{n_k} \in \left(\alpha - \frac{1}{k}, \alpha + \frac{1}{k}\right)$ so that $n_k > n_{k-1}$ (this can be done because there are infinitely many indices m with a_m in the interval).

3. (a) (2 pts) Give the formal definition for a series $\sum_{n=0}^{\infty} a_n$ being Cauchy.

Solution: $\forall \epsilon > 0$, $\exists N$ so that $\forall m > n > N$,

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon.$$

(b) (2 pts) Negate your answer to part (a) to give the formal definition for a series $\sum_{n=0}^{\infty} a_n$ not being Cauchy.

Solution: $\exists \epsilon > 0$ so that $\forall N$, $\exists m > n > N$ so that

$$\left| \sum_{k=n+1}^{m} a_k \right| \geq \epsilon.$$

(c) (6 pts) Prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by proving it is not Cauchy.

Hint: choose $\epsilon = 1/2$ and $m = 2n$.

Solution: Let $\epsilon = \frac{1}{2}$, and let N be given. Then choose any $n > N$ and choose $m = 2n$. Then

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \cdots + \frac{1}{2n} = n\frac{1}{2n} = \frac{1}{2},$$

so the series is not Cauchy, therefore not convergent.

4. (a) (3 pts) Give the formal definition of d being a metric on the space X.

   Solution: A metric is a function $d : X \times X \to \mathbb{R}$ satisfying:

   i. Non-negative: $d(x, y) \geq 0$ for all x, y.
   ii. Positive definite: $d(x, y) = 0$ iff $x = y$.
   iii. Symmetric: $d(x, y) = d(y, x)$ for all x, y.
   iv. Triangle Inequality: $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z.

   (b) (5 pts) Let $X = \mathbb{R} \times \mathbb{R}$ and let

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

   (i.e. the Euclidean metric). Define

$$d^*(x, y) = \min\{1, d(x, y)\}.$$

   Verify that d^* is a metric on X.

   Solution: see homework 6 solutions.

   (c) (2 pts) Draw neighborhoods with respect to d^* centered at the origin and of radius $\frac{1}{2}$, 1, and 2.

   Solution: The neighborhoods of radius $\frac{1}{2}$ and 1 are the same as those with respect to d: circles centered at the origin with radius $\frac{1}{2}$ and 1, respectively. However, the neighborhood of radius 2 is the entire space $\mathbb{R}^2$.

5. Let (X, d) be a metric space.

(a) (2 pts) Give the formal definition of a set $A \subseteq X$ being open. Define any terms you use such as 'neighborhood'.

Solution: A is open if $\forall x \in A$, $\exists \epsilon > 0$ so that $B_\epsilon(x) \subseteq A$. Note that $B_\epsilon(x) = \{y : d(x, y) < \epsilon\}$.

(b) (2 pts) Give the formal definition of a set $B \subseteq X$ being closed.

Solution: B is closed if $B^c = X \backslash B$ is open.

(c) (3 pts) Suppose $A \subseteq X$ is open, and $B \subseteq X$ is closed. Prove that $A \backslash B$ is open.

Solution: see homework 6 solutions.

(d) (3 pts) Prove that the set $\left\{x \in (0, 1) : x \neq \frac{1}{n} \text{ for every } n \in \mathbb{N}\right\}$ is open. Prove all your claims.

Solution: $(0, 1)$ is open. The set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ has only one limit point, 0. So the set $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closed. By part (c),

$$(0, 1) \backslash \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} = \left\{x \in (0, 1) : x \neq \frac{1}{n} \text{ for every } n \in \mathbb{N}\right\}$$

is open.

6. Let (t_n) be a bounded sequence of real numbers and let (s_n) be a sequence of real numbers that converges to 0.

   (a) (5 pts) Prove that $s_n t_n \to 0$ using only the definition of convergence (no theorems).

   Solution: see homework 3 solutions.

   (b) (5 pts) Assume that each $s_n > 0$. Prove that the sequence $\frac{1}{s_n}$ diverges to infinity. Use only the definition of divergence to infinity (no theorems).

   Solution: Since each $s_n > 0$, the fractions $\frac{1}{s_n}$ are well defined and all positive. Let $M > 0$ be a given natural number. Set $\epsilon = \frac{1}{M}$. Since $s_n \to 0$, $\exists N$ so that $\forall n > N$, $s_n < \epsilon$. Therefore, for $n > N$, $\frac{1}{s_n} > M$, so $\left(\frac{1}{s_n}\right)$ diverges to ∞.

7. Let $a_0 > b_0 > 0$ be fixed constants. Recursively define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$ for $n \geq 0$.

   (a) (4 pts) Use induction to prove $a_n > a_{n+1} > b_{n+1} > b_n$.

   Solution: Base step: $2a_0 > a_0 + b_0$ so $a_0 > a_1$. Also, $b_1 = \sqrt{a_0 b_0} > \sqrt{b_0^2} = b_0$. The hard inequality to show is the middle one. The claim can be written as $a_0 + b_0 > 2\sqrt{a_0 b_0}$, which means $a_0^2 + 2a_0b_0 + b_0^2 > 4a_0b_0$, which in turn can be written as $a_0^2 - 2a_0b_0 + b_0^2 = (a_0 - b_0)^2 > 0$ which is certainly true because $a_0 \neq b_0$.

   The inductive step is very similar.

   (b) (3 pts) Use your result from part (a) to explain why both (a_n) and (b_n) converge.

   Solution: (a_n) is bounded below by b_0 and monotone decreasing. Therefore it has a limit, call it a. (b_n) is bounded above by a_0 and monotone increasing. Therefore it has a limit, call it b.

   (c) (3 pts) Prove that $\lim a_n = \lim b_n$. This limit is called the arithmetic-geometric mean of a_0 and b_0.

   Solution: You can use either equation to get an equation for the limits. Using the definition of (a_n), we get $a = \frac{a+b}{2}$ which has solution $a = b$.