Homework 1 Solutions

The problems with a * were graded. This assignment is worth 20 points.

1. Section 1.1 (p. 8) # 3, 7, 10* (5 pts).

   Solution: The answer to # 3, 7 are in the back of your book, and the answer to # 10 is $x = 2, y = -1$.

2. Find a value of c and t so that the following system of equations has no solution:

   $$\begin{aligned} x + 2y &= 10 \\ 2x + cy &= t \end{aligned}$$

   Solution: Choose $c = 4$ and $t \neq 20$.

3. Section 1.2 (p. 19) # 5(a), 6(e), $T.7^*$ (5 pts).

   Solution: The answer to # 5(a) is in the back of your book. The answer to # 6(e) is

   $$\begin{pmatrix} 3 & 4 \\ 6 & 3 \\ 9 & 10 \end{pmatrix}.$$

   T.7 says to characterize the entries on the diagonal of $A - A^t$. You should notice that the diagonal entries of A^t are the same as the diagonal entries of A (because the i, jth entry of A^t is a_{ji}, so if $i = j$, you get back what you started with). Therefore, the diagonal entries of $A - A^t$ are all zero.

4. Characterize all the 2×2 matrices that can be written as a linear combination of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Solution: An arbitrary linear combination of these matrices is of the form:

   $$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ \mu & 0 \end{pmatrix}.$$

   Therefore, any 2×2 matrix A which is a linear combination of these matrices must satisfy the following properties:

   (a) $a_{22} = 0$.

   (b) $a_{21} = a_{12}$.

   (This means that the collection of 2×2 matrices which are linear combinations of these matrices has 2 'degrees of freedom', or is 2-dimensional. This is not surprising; generically, the set of linear combinations of n matrices will have dimension n... we'll discuss this later in the course.)

5. Let A be a matrix and λ a real number. Explain why $(\lambda A)^t = \lambda A^t$.

   Solution: Let $A = (a_{ij})$ be a matrix. Then $\lambda A = (\lambda a_{ij})$, so

$$(\lambda A)^t = (\lambda a_{ji}) = \lambda(a_{ji}) = \lambda A^t.$$

6. * (6 pts) Let $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   (a) Compute $v \cdot v$, and use this to normalize v.

   Solution: $v \cdot v = 2$, so the length of v is $\sqrt{2}$, and hence the normalization of v is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

   (b) Find a vector w so that $v \cdot w = 0$.

   Solution: Try $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

7. Section 1.3 (p. 34) # 7(a), 11* (4 pts).

   Solution: The answers to both these problems are in the back of the book.

8. Show that when A and B are diagonal $n \times n$ matrices, $AB = BA$.

   Note: to get started, write

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

   and write B similarly. Compute both AB and BA and compare them.

   Solution: Let A and B be diagonal $n \times n$ matrices, written as in the above note. Then

$$AB = \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 \\ 0 & a_2b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_nb_n \end{pmatrix},$$

   and

$$BA = \begin{pmatrix} b_1a_1 & 0 & \cdots & 0 \\ 0 & b_2a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_na_n \end{pmatrix},$$

   and since $a_ib_i = b_ia_i$ for all real numbers a_i and b_i, $AB = BA$.