Homework 2 Solutions

Chapter 1 problems

1. Verify the Law of Modus Ponens

$$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$$

is a tautology. (In practice, the Law of Modus Ponens means if we know P is true and we know $P \Rightarrow Q$ is true, then it's okay to assume Q is also true.)

Solution: One way to do this is with a truth table:

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

2. Prove or give a counterexample: for all natural numbers m, if m^2 is odd, then m must be odd.

Solution: Proof by contrapositive: Suppose m is even. Then $m = 2k$ for some integer k. Then $m^2 = (2k)^2 = 2(2k^2)$ so that m^2 is also even.

3. Section 1.5 # 11.

Solution: Proof by contradiction: Suppose $y - x \geq 1/2$ and $z - y \geq 1/2$. Then $z - x = z - y + y - x \geq 1$, a contradiction to the fact that x and z are both in the open unit interval.

4. Prove: for all real numbers $\epsilon > 0$, there exists a natural number N so that $\frac{1}{N^3} < \epsilon$.

Solution: Let $\epsilon > 0$ be given. Choose N to be a natural number larger than $1/\sqrt[3]{\epsilon}$. Then

$$N > \frac{1}{\sqrt[3]{\epsilon}}$$
$$\frac{1}{N} < \sqrt[3]{\epsilon}$$
$$\frac{1}{N^3} < \epsilon$$

5. Section 1.7 # 2(b)(c).

   Solution: 2(b): The x-axis has slope 0, whereas l has slope $-2/k$ which is never zero.
   2(c): Any line of the form of l passing through the point $(1, 4)$ must have $k = -2$. Therefore, there's a unique such line.

Chapter 2 problems

6. Define the set $X_a = \{x : x \in \mathbb{Z} \text{ and } x^2 < a\}$ for each $a \in \mathbb{N}$.

   (a) How many elements does X_{12} contain? List them.
   Solution: $X_{12} = \{-3, -2, -1, 0, 1, 2, 3\}$ so that X_{12} has 7 elements.

   (b) For which values of a is the equation $X_a = X_{12}$ true?
   Solution: $X_a = X_{12}$ for $a \in \{10, 11, 12, 13, 14, 15, 16\}$.

   (c) Find the intersection $\bigcap_{a\in\mathbb{N}} X_a$ and the union $\bigcup_{a\in\mathbb{N}} X_a$.

   Solution: For any number $x \in \mathbb{Z}$, $x \in X_{|x|^2+1}$, so that $\bigcup_{a\in\mathbb{N}} X_a = \mathbb{Z}$. On the other hand, the only element common to all X_a is zero, so $\bigcap_{a\in\mathbb{N}} X_a = \{0\}$.

7. Section 2.1 # 8(a)(c), 11.

   Solution: 8(a): Any set with 6 elements, such as $A = \{1, 2, 3, 4, 5, 6\}$.

   8(c): No set A has the property that $\mathcal{P}(A) = \emptyset$. See worksheet 2.

   11: Suppose by contradiction that $x \notin B$, $A \subseteq B$, but $x \in A$. Then since $A \subseteq B$, $x \in B$. But this contradicts $x \notin B$. Therefore all $x \notin B$ are also not in A.

8. Section 2.2 # 3(a)(i)(l)(m), 13(b)(c), 14(b).

   Solution: 3(a): $[2, 8)$.

   3(i):$[2, 5]$.

   3(l): $(-\infty, 5]$.

   3(m): $\emptyset$.

   13(b): Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. Thus, $X \subseteq A$ or $X \subseteq B$. Therefore, $X \subseteq A \cup B$. Finally, this means $X \in \mathcal{P}(A \cup B)$.

   13(c): If the sets A and B are disjoint, then equality is not obtained. For example, if $A = \{0\}$ and $B = \{1\}$, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{0\}, \{1\}\}$, but $\mathcal{P}(A \cup B) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. If $A \subseteq B$ or $B \subseteq A$, then equality is obtained.

   14(b): Let $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{1\}$.

9. Let A and B be sets. Prove that $A \setminus B = A \cap B^c$.

Solution:

$$\begin{aligned} x \in A \setminus B &\Leftrightarrow x \in A \,\text{and}\, x \notin B \\ &\Leftrightarrow x \in A \,\text{and}\, x \in B^c \\ &\Leftrightarrow x \in A \cap B^c \end{aligned}$$