Homework 2 Solutions

The problems with a * were graded (1.4 # 15, 1.5 # 19(a), # 5, and # 7(b)). All are worth 5 points, so the total for this assignment is 20 points.

1. Section 1.4 (p. 49) # 15*, T.9.

   Solution: The answer to # 15 is in the back of your book, $r = 3$. Note that means that 3 is an **eigenvalue** of the matrix A, and $x = (1, 1)$ is an **eigenvector**. We'll discuss these concepts in detail later in the course.

   For T.9, you are asked to find a matrix that commutes with A. If you take an arbitrary B and work out the algebra for the equation $AB = BA$, you'll get the restrictions $b_{11} = b_{22}$, $b_{21} = 0$, and no restriction on b_{12}. An example of such a matrix is
   $$\begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix}.$$
   There are infinitely many such matrices B. (We had two 'degrees of freedom' in our characterization, so the set of matrices commuting with A is 2-dimensional.)

2. Write an equation for the matrix transformation which takes the vector (w, x, y, z) to the vector $(2w + x, y - z)$.

   Solution:
   $$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

3. Section 1.5 # 10 (explicitly write the pre-image of the vector w), 19(a)*.

   Solution: For # 10, the pre-image of the vector w is $(1, 1)$. For # 19(a),
   $$A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$
   so
   $$A^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$
   which is the matrix for rotation by $2\pi/3$.

   In more generality: let R_θ be the rotation matrix by θ degrees. Then $R_\theta R_\phi = R_{\theta+\phi}$. Try to prove this yourself (I'll probably put it on a quiz or midterm). You may need to use the identities: $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.

4. Section 1.6 # 21(a)(b).

   Solution: Both are in the back of your book.

5. * Find the kernel of the matrix transformation given by $f(x) = Ax$, where

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

Solution: Setting $Ax = 0$, where $x = (x_1, x_2, x_3)$, we find that $(x_1 - x_2, x_2 - 2x_3) = (0, 0)$. Therefore, $x_1 = x_2$ and $x_2 = 2x_3$. Therefore, there is 1 degree of freedom in the kernel (i.e. the kernel is one-dimensional). x_1 can be any real number, but then $x_2 = x_1$ and $x_3 = \frac{1}{2}x_1$. Formally,

$$\ker A = \left\{ \left(r, r, \frac{1}{2}r \right) \text{ where } r \text{ is any real number} \right\}.$$

6. Suppose A and B are both 2×2 square matrices. The Kronecker product (or tensor product) of two such matrices is the 4×4 matrix defined by

$$A \otimes B = \left(\begin{array}{c|c} a_{11}B & a_{12}B \\ \hline a_{21}B & a_{22}B \end{array} \right)$$

Suppose

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Compute $A \otimes A$.

Solution:

$$A \otimes A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

(Application: the matrix A comes up a lot in quantum mechanics. It can be thought of as a transformation on a single photon. In quantum mechanics, if you have a transformation A on a photon and a transformation B on another photon, then oftentimes the transformation on both photons together is given by $A \otimes B$.)

7. (a) Explain why the adjacency matrix of an undirected graph is always symmetric.

   Solution: A symmetric matrix is one where $A^t = A$, so that for each pair i, j, $a_{ij} = a_{ji}$. In an adjacency matrix, the number a_{ij} is 1 if there is an edge between vertices i and j and it's zero otherwise. Certainly, if there's an edge between vertices i and j, then there's an edge between vertices j and i, so $a_{ij} = a_{ji} = 1$. Otherwise, if there's no edge between i and j, then there's no edge between j and i, so $a_{ij} = a_{ji} = 0$. In either case, the entries are equal, so the matrix is symmetric.