Homework 2 Solutions

1. (1.10) Prove $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1)=3n^2$ for all $n \in \mathbb{N}$, $n \geq 1$.

   Solution: Base step: When $n=1$, we get $3=3$, which is true.

   Inductive step: Suppose $(2k+1)+(2k+3)+(2k+5)+\cdots+(4k-1)=3k^2$. We'll prove it holds for $k+1$.

$$\begin{aligned}(2(k+1)+1)+\cdots+(4(k+1)-1) &= (2k+3)+\cdots+(4k-1)+(4k+1)+(4k+3) \\ &= 3k^2-(2k+1)+(4k+1)+(4k+3)=3(k+1)^2\end{aligned}$$

2. Use induction to prove Bernoulli's inequality:

$$\text{If } 1+x>0, \text{ then } (1+x)^n \geq 1+nx \text{ for all } n \in \mathbb{N}.$$

   Solution: Base step: When $n=0$, we get $1=1$, which is true.

   Inductive step: Suppose $(1+x)^k \geq 1+kx$.

$$\begin{aligned}(1+x)^{k+1}=(1+x)(1+x)^k &\geq (1+x)(1+kx) \\ &= 1+(k+1)x+kx^2 \geq 1+(k+1)x\end{aligned}$$

3. **Polya's paradox.** Find the error in the following proof by induction.

   Claim: All horses are the same color.
   Proof:
   Base case: If there's only one horse, then all horses are the same color.
   Inductive step: Suppose that any set of n horses have the same color. Now suppose we have a set of $n+1$ horses. Take all but the first horse, i.e. horses $\{2,3,\ldots,n+1\}$. This is a set of n horses, so they each have the same color by the inductive hypothesis. Now, take all but the last horse, i.e. horses $\{1,2,\ldots,n\}$. This is also a set of n horses, so by the inductive hypothesis they each have the same color. Because the sets overlap at horses $\{2,3,\ldots,n\}$, both sets of horses are all the same color. Therefore, by induction every horse is the same color.

   Solution: If $n=2$, then the two sets don't overlap. This "paradox" is meant to illustrate that sometimes there are special cases that must be proved separately, like $n=2$ here. Since we cannot prove all pairs of horses have the same color, the proof does not work (as it shouldn't, because the claim that all horses are the same color is false!). If you're interested in reading more about this, see:

   `http://en.wikipedia.org/wiki/All_horses_are_the_same_color`

4. Prove that $ac = bc$ and $c \neq 0$ imply $a = b$ for $a, b, c \in \mathbb{Q}$. Write down each field property (A1-A4, M1-M4, DL) or order property (O2-O6) each time you use it.

   Solution:

$$
\begin{aligned}
ac = bc &\Rightarrow ac + (-bc) = bc + (-bc) \text{ by A4} \\
&\Rightarrow ac + (-bc) = 0 \text{ by A4} \\
&\Rightarrow ca + (-cb) = 0 \text{ by M2} \\
&\Rightarrow ca + c(-b)) = 0 \text{ by Theorem 3.1 (iii)} \\
&\Rightarrow c(a + (-b)) = 0 \text{ by DL} \\
&\Rightarrow a + (-b) = 0 \text{ by Theorem 3.1 (vi) and the assumption that } c \neq 0 \\
&\Rightarrow [a + (-b)] + b = 0 + b = b \text{ by A3} \\
&\Rightarrow a + [(-b) + b] = a + 0 = a = b \text{ by A1, A4, and A3.}
\end{aligned}
$$

5. Prove that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{Q}$. Write down each field property (A1-A4, M1-M4, DL) or order property (O2-O6) each time you use it.

   Solution: I'm NOT going to cite each field and order property this time, but rather give you the idea of the proof. For you to get full credit, however, you must cite each field and order property.

   We want to show that $||a| - |b|| \leq |a - b|$, which is the same thing as $-|a - b| \leq |a| - |b| \leq |a - b|$. Now, $|a| = |a - b + b| \leq |a - b| + |b|$ by the triangle inequality, so $|a| - |b| \leq |a - b|$, which established one of the inequalities we need to show. To establish the other, merely reverse the role of a and b: $|b| = |a + b - a| \leq |a| + |a - b|$ again by the triangle inequality and the fact that $|x - y| = |y - x|$. Therefore, $-|a - b| \leq |a| - |b|$, which is the other inequality we needed to show.

6. Prove: if $0 \leq x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.

   Solution: Suppose by contradiction that $x > 0$. Then set $\epsilon = x/2$. Then $\epsilon > 0$ yet $x > \epsilon$, which contradicts the assumption that $x \leq \epsilon$ for all $\epsilon > 0$.

7. (4.14) Let A and B be non-empty subsets of $\mathbb{R}$ that are bounded above, and let

$$S = \{a + b : a \in A, b \in B\}.$$

   Prove that $\sup S = \sup A + \sup B$.

   Solution: Let $\alpha = \sup A$ and $\beta = \sup B$. We claim that $\alpha + \beta = \sup S$. First of all, any element in S is of the form $a + b \leq \alpha + \beta$ by definition of α and β. So, $\alpha + \beta$ is an upper bound for S. To prove that it is the supremum we need to show that $\alpha + \beta$ is the least upper bound. That is, if δ is an upper bound of S then we need to show that $\alpha + \beta \leq \delta$. We argue by contradiction: suppose there was some upper bound δ of S so that $\alpha + \beta > \delta$. Let $d = (\alpha + \beta) - \delta$. Then since $d > 0$, $\alpha - d/2$ and $\beta - d/2$ cannot be upper bounds for A and B, respectively.

Therefore, $\exists a \in A$, and $\exists b \in B$ so that $a > \alpha - d/2$ and $b > \beta - d/2$. But then we obtain a contradiction to δ having been an upper bound for S:

$$a + b > \alpha - d/2 + \beta - d/2 = \alpha + \beta - d = \delta$$

8. (4.16) Show that for all $a \in \mathbb{R}$, $\sup\{r \in \mathbb{Q} : r < a\} = a$.

   Solution: Clearly a is an upper bound for the set by construction. Now suppose there is some other upper bound, call it x. Then we need to explain why $a \leq x$ (so that a is the least upper bound). Suppose by contradiction that $a > x$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there is some rational q so that $a > q > x$. However, if $q < a$, q must be in our set. Therefore, x cannot be an upper bound.

9. In this exercise you will prove the irrational numbers are dense in $\mathbb{R}$.

   (a) Let $x \in \mathbb{Q}$ and $x \neq 0$. Let $y \in \mathbb{R} \setminus \mathbb{Q}$ (i.e. y is irrational). Prove that xy is irrational.

   Solution: Since x is rational, $x = m/n$ for some integers m, n. Suppose that y is irrational, and suppose by contradiction that $xy = p/q$ is rational. We'll contradict the fact that y is irrational by writing it as a fraction:

$$y = \frac{xy}{x} = \frac{p/q}{m/n} = \frac{pn}{mq}.$$

   (b) Let $x, y \in \mathbb{R}$ so that $x < y$. Prove that there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ so that $x < w < y$. Hints: apply the density of $\mathbb{Q}$ in $\mathbb{R}$ to $x/\sqrt{2}$ and $y/\sqrt{2}$ and then use part (a).

   Solution: By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists some number $q \in \mathbb{Q}$ so that

$$\frac{x}{\sqrt{2}} < q < \frac{y}{\sqrt{2}}.$$

   Then, mutlplying everything by $\sqrt{2}$, set $z = \sqrt{2}q$. By part (a), z is irrational and lies between x and y. Note: in this proof I assume $\sqrt{2}$ is irrational, which we never actually proved. Try proving this yourself or look it up on Wikipedia:

   `http://en.wikipedia.org/wiki/Square_root_of_2#Proofs_of_irrationality`