Homework 3 Solutions

The problems with a * were graded. Problem 1.7 # 6(b) and 3.1 # 17(b) are worth 5 points, and problem 5 is worth 7 points (so the total is 17 points).

1. Section 1.7 # 6(a)(b)*.

   Solution: (a): the matrix is a not invertible. (b): the inverse is:

$$\begin{pmatrix} 1 & -1 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ -1 & 0 & 1 \end{pmatrix}.$$

2. A matrix A is **orthogonal** if it satisfies the equation $A^tA = AA^t = I$ (in other words, $A^{-1} = A^t$). Show that the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   is orthogonal for any choice of θ.

   Solution:

$$R_\theta R_\theta^t = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = I.$$

   The only identity used is $\cos^2\theta + \sin^2\theta = 1$.

3. Section 3.1 # 17(b)*, T.6.

   Solution: The answer to 17(b) is in the back of your book (it's zero). For T.6, the answer is 'yes'. Here's why: $\det(AB) = \det(A)\det(B)$ and $\det(BA) = \det(B)\det(A)$ by one of the properties of the determinant map. Since $\det(A)$ and $\det(B)$ are real numbers, they commute, i.e. $\det(A)\det(B) = \det(B)\det(A)$, so $\det(AB) = \det(BA)$.

4. Section 3.2 # 6(a), 10(b).

   Solution: The answer to 6(a) is -135. The answer to 10(b) is:

$$\frac{1}{14}\begin{pmatrix} 3 & -9 & 2 \\ 2 & 10 & -8 \\ -1 & 2 & 4 \end{pmatrix}.$$

5. *Recall that the projection of a vector $v = (v_1, v_2)$ onto the vector $u = (u_1, u_2)$ is the vector

$$\left(\frac{u \cdot v}{\|u\|^2}\right) u = \frac{u_1v_1 + u_2v_2}{u_1^2 + u_2^2}(u_1, u_2). \tag{1}$$

For example, the projection of the vector $(1,2)$ onto the vector $(1,1)$ is the vector

$$\frac{3}{2}(1,1) = \left(\frac{3}{2}, \frac{3}{2}\right).$$

Use equation (1) to write the matrix tranformation that projects a vector in $\mathbb{R}^2$ onto the line $y = 3x$.

(Hint: let $u = (u_1, u_2) = (1,3)$; it 'generates' the line $y = 3x$.)

Solution: The projection of the vector (v_1, v_2) onto the line $y = 3x$ results in the vector $\frac{1}{10}(v_1 + 3v_2, 3v_1 + 9v_2)$. The matrix which computes this transformation is

$$P = \frac{1}{10}\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}.$$

Note that $P^2 = P$, as it should!

Also, you could have used any vector for u which lies on the line $y = 3x$, such as the vector $(1/2, 3/2)$ or $(2,6)$. As an exercise, you might check that you get the same answer for a few different such choices of u.

The problem can be made more general by not specifying u. If $u = (u_1, u_2)$ is left arbitrary, then the matrix transformation is:

$$\frac{1}{u_1^2 + u_2^2}\begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix}.$$

Check that this is an idempotent. Conversely, given an arbitrary idempotent, can you figure out what line it projects onto? Another question: the projection matrix above is symmetric; is it always true that idempotent matrices are symmetric? (I would recommend thinking about these questions before the midterm.)