Homework 4 Solutions

Grading: Problems 1(a), 2, 6(a), each worth 5 points (total 15 points).

1. Let A be an $n \times n$ matrix. A **square root** of A is an $n \times n$ matrix B so that $B^2 = A$.

   (a) *Find a square root of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

   Solution: Let
   $$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
   Calculate B^2, and set this equal to A. You'll get the system of equations
   $$\begin{cases} a^2 + bc = 1 \\ ab + bd = 1 \\ ac + dc = 0 \\ bc + d^2 = 1 \end{cases}$$
   The third equation can be written $c(a + d) = 0$. This is true if $c = 0$ or $a+b = 0$. Let's try $c = 0$. Then this reduces the other equations to $a^2 = 1$, $ab + bd = 1$, and $d^2 = 1$. Trying out $a = d = 1$, we obtain $2b = 1$, so $b = 1/2$. Plugging this in and checking, we obtain a solution
   $$B = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$
   Notice that we had different options throughout the calculation. For example, we could have chosen $a = d = -1$, and then $b = -1/2$. This matrix is a scalar multiple of the first example. Are there other examples? Does the set of square roots of A form a subspace of M_2?

   (b) Show that there is no square root of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   Solution: In this example we get the system of equations
   $$\begin{cases} a^2 + bc = 0 \\ ab + bd = 1 \\ ac + dc = 0 \\ bc + d^2 = 0 \end{cases}$$
   The third equation tells me again $c = 0$ or $a + d = 0$. However, if $c = 0$, then $a^2 = d^2 = 0$ by the first and fourth equations, so $a = d = 0$. But then the second equation gives us $0 = 1$, which is a contradiction. So, c can't be zero. Then it must be the other case, $a = -d$. But then the second equation again gives us $0 = 1$, which is a contradiction. So both cases are impossible. Therefore, there is no square root.

2. *Recall that an **idempotent** matrix A satisfies the relation $A^2 = A$. Find the determinant of an idempotent matrix.

   Solution: Since $A^2 = A$, it must be the case that $\det(A^2) = \det(A)$. Since the determinant map respects multiplication, we can rewrite this as $\det(A)^2 = \det(A)$. You can check that the only solutions to this equation are $\det(A) = 0$ or $\det(A) = 1$.

3. Let $\mathbb{R}_+ = \{x \text{ in } \mathbb{R} \mid x > 0\}$. Explain why $\mathbb{R}_+$ is NOT a vector space. List all the axioms for a vector space that it does not satisfy (only consider the ones I starred in class).

   Solution: The only axiom $\mathbb{R}_+$ satifies is closure under addition, because the sum of two positive numbers is always positive. $\mathbb{R}_+$ does not contain zero by definition, it doesn't contain inverses because the inverse of a positive number is negative, and it doesn't contain scalar multiplication by negative numbers.

4. Section 6.2 # 5.

   Solution: (a) is not a subspace because it doesn't even contain zero. (b) is a subspace, which is relatively easy to check. (c) is not a subspace, again it does not even contain zero.

5. Let M_n be the set of all $n \times n$ matrices with real entries. Let S_n be the subset of M_n consisting of symmetric matrices.

   (a) Show that

   $$S_2 = \operatorname{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

   Solution:

   $$\operatorname{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\right\} = S_2.$$

   (b) Find matrices $A_1, A_2, \ldots, A_6$ so that $S_3 = \operatorname{span}\{A_1, A_2, \ldots A_6\}$. Can you generalize this to arbitrary n?

   Solution: Let A_1, A_2, A_3 be the matrices with a 1 in the first, second, and third diagonal entry, respectively (and all zeroes otherwise). Let A_4 have a one in the a_{12} and a_{21} spots, and zero everywhere else; A_5 have a one in the a_{13} and a_{31} spots, and zero everywhere else; and A_6 have a one in the a_{23} and a_{32} spots, and zero everywhere else.

   For any arbitrary n, there need to be $\frac{n(n+1)}{2}$ elements in the span. These consist of the n matrices that have a one somewhere on the diagonal, and zeroes everyone else, and the $\frac{n(n-1)}{2}$ matrices that have a one in the i, jth and j, ith spots simulatenously, and zeroes elsewhere.

6. The **trace** of an $n \times n$ matrix A is the sum of its diagonal entries:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Let M_n be the set of all $n \times n$ matrices with real entries. Let M_n^0 be the subset of M_n consisting of matrices with trace zero.

(a) *Directly show that M_n^0 is a subspace of M_n.

Solution: $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ because $\sum_{i=0}^{n} a_{ii} + b_{ii} = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$. Therefore, M_n^0 is closed under addition. Also, since $\sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii}$, M_n^0 is closed under scalar multiplication. (The above formulas only imply closure of addition and scalar multiplication when we consider trace zero. For example, if the traces of two matrices A and B are 2 then the trace of $A + B$ would be 4.) M_n^0 contains zero and inverses because these are multiplication by 0 and -1, respectively.

(b) Write M_2^0 as the span of three matrices in M_2. In other words, find matrices A_1, A_2, A_3 in M_2 so that

$$M_2^0 = \operatorname{span}\{A_1, A_2, A_3\}.$$

Can you generalize this to arbitrary n?

Solution: An arbitrary 2×2 matrix with trace zero has the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

which can be written as a linear combination of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so these elements span the 2×2 matrices with trace zero.

For arbitrary n, there need to be $n^2 - 1$ elements in the span. They consist of all the $n^2 - n$ matrices with a one somewhere off the diagonal, and zeroes everywhere else, and $n - 1$ matrices that have a one somewhere on the diagonal and a -1 in the last diagonal slot.