Homework 4 Solutions

1. For each of the following sequences (s_n), find the set S of subsequential limits, and use this to find $\liminf s_n$ and $\limsup s_n$.

    (a) (11.1) $s_n = 3 + 2(-1)^n$.

    Solution: $(s_n) = (5, 1, 5, 1, \dots)$, so $S = \{1, 5\}$, and therefore $\limsup s_n = 5$ and $\liminf s_n = 1$.

    (b) $s_n = n^2(-1 + (-1)^n)$.

    Solution: $(s_n) = (0, -2, 0, -18, 0, -50, \dots)$, so $S = \{-\infty, 0\}$, and therefore $\limsup s_n = 0$ and $\liminf s_n = -\infty$.

    (c) $(s_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \dots)$.

    Solution: $S = \{0, 1, \infty\}$, so $\limsup s_n = \infty$ and $\liminf s_n = 0$.

2. Prove that every unbounded sequence contains a monotone subsequence that diverges to either ∞ or $-\infty$.

    Solution: If (s_n) is unbounded above, for each choice of K, there exists a sequence element s_{n_k} so that $s_{n_k} > K$. The subsequence (s_{n_k}) clearly diverges to ∞. Proof is similar if (s_n) is unbounded below.

3. Suppose $x > 1$. Prove that $\lim x^{1/n} = 1$.

    Solution: First of all, $x^{1/n}$ is bounded below by 1 since $x > 1$. Note the quantity $x^{\frac{1}{n(n+1)}} > 1$, so

    $$x^{\frac{1}{n+1}} < x^{\frac{1}{n(n+1)}} x^{\frac{1}{n+1}} = x^{\frac{1}{n}},$$

    which means the sequence is decreasing. Since it is decreasing and bounded below by 1, a limit exists for the sequence, call it s. Note any subsequence must converge to s. However, the subsequence $x^{1/2n}$ converges to $\sqrt{s}$, so we have $s = \sqrt{s}$. Therefore, $s = 0$ or $s = 1$. Since $x^{1/n} > 1$ for all n, it must be the case that $s = 1$.

4. Suppose that (s_n) and (t_n) are bounded. Prove that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$. Find an example which shows they are not equal.

    Solution: To prove the inequality, it's sufficient to prove that the inequality holds for any convergent subsequence. In other words, let $(s_{n_k} + t_{n_k})$ be a convergent subsequence of $(s_n + t_n)$, and denote its limit by a.

    Let $s = \limsup s_n$ and $t = \limsup t_n$. We want to show $a \leq s + t$. We'll show that $\forall \epsilon > 0$, $a \leq s + t + \epsilon$.

So, let $\epsilon > 0$ be given. Then for $\epsilon/2$, there exists N_1 so that $\forall n > N_1$,

$$\sup\{s_n : n > N_1\} < s + \epsilon/2,$$

so for $n > N_1$, $s_n < s + \epsilon/2$. Similarly, $\exists N_2$ so that $t_n < t + \epsilon/2$ for $n > N_2$. Then for $n_k > \max\{N_1, N_2\}$, we have

$$s_{n_k} + t_{n_k} < s + \epsilon/2 + t + \epsilon/2 = s + t + \epsilon$$

So then $a = \lim s_{n_k} + t_{n_k} \leq s + t + \epsilon$.

For an example where they're not equal, let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$. Then $s_n + t_n = 0$, so $\limsup(s_n + t_n) = 0$, but $\limsup s_n + \limsup t_n = 1 + 1 = 2$.

5. (12.2) Prove that $\limsup |s_n| = 0$ iff $\lim s_n = 0$.

   Solution: If $\limsup |s_n| = 0$, then $\forall \epsilon > 0$, $\exists N$ so that $\sup\{|s_n| : n > N\} < \epsilon$. By the nature of sup, this means $|s_n| < \epsilon$ for all $n > N$. Therefore $s_n \to 0$. The argument can be easily reversed for the other direction.

6. In this exercise you'll prove that the sequence (s_n) defined by

$$s_n = \left(1 + \frac{1}{n}\right)^n$$

   for $n \geq 1$ converges. Its limit is the number e, which is approximately 2.71828.

   (a) Prove by induction $b^{n+1} - a^{n+1} = (b-a)(b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n)$ for $n \geq 1$.

   Solution: When $n = 1$, we get $b^2 - a^2 = (b-a)(b+a)$, which is true. Now suppose the result holds for $k - 1$, i.e.

$$b^k - a^k = (b-a)(b^{k-1} + b^{k-2}a + \cdots + a^{k-1}).$$

   Then

$$\begin{aligned} (b-a)(b^k + b^{k-1}a + \cdots ba^{k-1} + a^k) &= (b-a)(b^k + a(b^{k-1} + \cdots + a^{k-1})) \\ &= b^k(b-a) + a(b^k - a^k) \\ &= b^{k+1} - ab^k + ab^k - a^{k+1} = b^{k+1} - a^{k+1}. \end{aligned}$$

   (b) Suppose $0 < a < b$. Prove $b^{n+1} - a^{n+1} < (n+1)b^n(b-a)$.

   Solution: Since $a < b$, for every k, $a^k < b^k$. Therefore,

$$b^k + ab^{k-1} + \cdots + a^{k-1}b + a^k < b^k + b^k + \cdots + b^k = (k+1)b^k.$$

   Therefore,

$$b^{n+1} - a^{n+1} = (b-a)(b^n + ab^{n-1} + \cdots + a^{n-1}b + a^n) < (b-a)(n+1)b^n.$$

(c) Subsitute $a = 1 + \frac{1}{n+1}$ and $b = 1 + \frac{1}{n}$ into the formula in part (b). Use this to show the sequence (s_n) is increasing, i.e. $s_{n+1} > s_n$ for all $n \geq 1$.

Solution: Part (b) says that

$$\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1} < (n+1)\left(1+\frac{1}{n}\right)^{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Therefore,

$$\left(1+\frac{1}{n}\right)^{n+1} - (n+1)\left(1+\frac{1}{n}\right)^{n}\left(\frac{1}{n}-\frac{1}{n+1}\right) < \left(1+\frac{1}{n+1}\right)^{n+1}.$$

$$\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}-(n+1)\left(\frac{1}{n}-\frac{1}{n+1}\right)\right) = \left(1+\frac{1}{n}\right)^{n} < \left(1+\frac{1}{n+1}\right)^{n+1},$$

which is what we wanted to prove.

(d) Prove that the subsequence of even terms is bounded by 4, i.e.

$$\left(1+\frac{1}{2n}\right)^{2n} \leq 4.$$

Do this by plugging $a = 1$ and $b = 1 + \frac{1}{2n}$ into the formula from part (b). Argue that since the sequence is increasing all terms must be bounded by 4.

Solution: Plugging in $a = 1$ and $b = 1 + \frac{1}{2n}$ we get

$$\left(1+\frac{1}{2n}\right)^{n+1} - 1 < (n+1)\left(1+\frac{1}{2n}\right)^{n}\frac{1}{2n}.$$

Rearranging,

$$\left(1+\frac{1}{2n}\right)^{n+1} - \frac{n+1}{2n}\left(1+\frac{1}{2n}\right)^{n} < 1$$

$$\left(1+\frac{1}{2n}\right)^{n}\left(1+\frac{1}{2n}-\frac{n+1}{2n}\right) < 1$$

$$\left(1+\frac{1}{2n}\right)^{n} < 2$$

$$\left(1+\frac{1}{2n}\right)^{2n} < 4$$

Therefore, all the even terms are bounded by 4. Because the sequence is increasing, any odd term is bounded above by some even term, which is bounded by 4. So all the terms are bounded by 4.

(e) Conclude the sequence converges, and denote its limit by e.

(f) Find the limits of

$$\left(1+\frac{1}{2n}\right)^{2n}, \quad \left(1+\frac{1}{n}\right)^{2n}, \quad \text{and} \quad \left(1+\frac{1}{2n}\right)^{n}.$$

Solution: The first sequence is the subsequence of all even terms, so it has the same limit as the original sequence:

$$\left(1+\frac{1}{2n}\right)^{2n} \to e.$$

The second sequence is the square of the original sequence, so its limit is:

$$\left(1+\frac{1}{n}\right)^{2n} \to e^2.$$

The third sequence is the square root of the sequence of even terms, so its limit is:

$$\left(1+\frac{1}{2n}\right)^{n} \to \sqrt{e}.$$