Homework 5 Solutions

1. Section 4.2 # 1(g), 3(e), 14(d).

   Solution: 1(g): $(f \circ g)(x) = \frac{x^2+2}{x^2+3}$, and $(g \circ f)(x) = \left(\frac{x+1}{x+2}\right)^2 + 1$.

   3(e), 14(d): answers in the back of the book.

2. Let $A = \mathbb{N} \cup \{0\}$. Define the equivalence relation $R_n \subseteq A \times A$ by $(x, y) \in R_n$ iff $x \mod n \equiv y \mod n$. Denote the set of equivalence classes $A/R_n = \mathbb{Z}_n$.

   (a) Let $f : \mathbb{Z}_4 \to \mathbb{Z}_2$ be the map given by $f(x/R_4) = (x \mod 2)/R_2$. Prove f is a well-defined function. Is f injective and/or surjective? Prove your claims.

   Solution: Suppose $x/R_4 = y/R_4$. Then $x \mod 4 \equiv y \mod 4$, which means $\exists k, j, r \in A$ so that $x = 4k + r$ and $y = 4j + r$. Since 4 is divisible by 2, $4k$ and $4j$ are $0 \mod 2$, so that $x \mod 2 \equiv y \mod 2 = r \mod 2$. Therefore, $f(x/R_4) = f(y/R_4)$ so that f is a well-defined function.

   f is not injective because $1/R_4 \neq 3/R_4$, yet $f(1/R_4) = f(3/R_4) = 1/R_2$. f is surjective though, because $f(0/R_4) = 0/R_2$ and $f(1/R_4) = 1/R_2$, so both elements in the codomain have a pre-image.

   (b) Let $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ be the map given by $f(x/R_4) = (x+1 \mod 4)/R_4$. Prove f is a well-defined function. Is f injective and/or surjective? Prove your claims.

   Solution: Suppose $x/R_4 = y/R_4$. Then $x \mod 4 \equiv y \mod 4$, which means $\exists k, j, r \in A$ so that $x = 4k + r$ and $y = 4j + r$. Then $x + 1 \mod 4 \equiv y + 1 \mod 4 = r + 1 \mod 4$. Therefore, $f(x/R_4) = f(y/R_4)$ so that f is a well-defined function.

   Suppose $(x+1)/R_4 = (y+1)/R_4$. Then $\exists k, j, r \in A$ so that $x+1 = 4k+r$ and $y + 1 = 4j + r$. Then, $x = 4k + r - 1$ and $y = 4j + r - 1$, so that $x \mod 4 \equiv y \mod 4 \equiv r - 1 \mod 4$. Therefore, $x/R_4 = y/R_4$. This means that f is injective.

   Since each x/R_4 has pre-image $(x - 1)/R_4$, the map is surjective.

3. Recall the kernel of a map $f : A \to B$ (where $0 \in B$) is the set

   $$ker(f) = \{x \in A \mid f(x) = 0\}.$$

   (a) Prove: if f is a bijection, then the kernel of f contains exactly one element.

   Solution: Because f is surjective, 0 must have a pre-image so there exists at least one element in $ker(f)$. Since f is injective, no two distinct elements can both map to 0, so $ker(f)$ can only have at most one element in it. Putting both these statements together, we see that $ker(f)$ must have exactly one element in it.

(b) Give a counterexample to show the converse of 3a does not hold.

Solution: Let $A = \{0, 1, 2\}$ and $B = \{0, 2\}$. Then the function $f = \{(0, 0), (1, 2), (2, 2)\}$ has a single element in its kernel, but it's not injective, so it's not surjective.

(c) Use the contrapositive of 3a to prove that $f(x) = x^2 - 5x + 6$ is not a bijection.

Solution: The contrapositive of 3a is: if $ker(f)$ has no elements, or has more than one element, then f is not a bijection. Since the function $f(x) = x^2 - 5x + 6$ has two elements in the kernel (the elements 2 and 3), it cannot be a bijection.

4. Let $A = \{1, \dots n\}$, and $f : A \to A$ be a surjective function. Prove f is also an injective function. Give a counterexample to show that surjectivity does not imply injectivity when A contains infinitely many elements.

   Solution: Since f is surjective, $ran(f) = A$, so that $ran(f)$ has n elements. Suppose f is not injective. Then, WLOG $f(a_1) = f(a_2)$. Then the range of f is given by $\{f(a_2), \dots, f(a_n)\}$, which has at most $n - 1$ elements. This is a contradiction.

   Define the function $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ by $f(k) = |k|$. Then f is surjective onto $\mathbb{N} \cup \{0\}$ but it is not injective. For example, $f(2) = f(-2) = 2$.

5. Section 4.3 # 8(a)(e), 9(a), 16(a)(e)(g).

   Solution: 8(a)(e): answers in the back of the book.

   9(a): f is not surjective, because 0 has no pre-image. This is true because $2 - x \geq 1$ if $x \leq 1$, and $1/x > 0$ for $x > 1$. However, f is injective. If $x, y \leq 1$, then $f(x) = 2 - x = 2 - y = f(y)$ implies $x = y$. If $x, y > 1$, then $1/x = 1/y$ implies $x = y$. Finally, if $x \leq 1$, then $f(x) = 2 - x \geq 1$, and if $y > 1$, then $f(y) = 1/y < 1$, so that $f(x) \neq f(y)$.

   16(a): Each element in the domain needs a unique element in the range to map to. For the first element we have n choices, for the second element we have $n - 1$, and so on. Since there are m elements in the domain, this totals $n(n - 1) \dots (n - m + 1)$.

   16(e)(g): Both these answers are the same, because when the domain and codomain are finite, the number of injective functions equals the number of surjective functions equals the number of bijective functions. Plugging $m = n$ into the answer for 16(a) gives the answer $n!$.

6. Give an example of a function $f : A \to B$ with the property that there exists a subset $C \subseteq B$ with $f(f^{-1}(C)) \neq C$.

   Solution: Let $A = \{0, 1, 2\}$, $B = \{0, 1, 2\}$, and $f = \{(0, 0), (1, 0), (2, 1)\}$. Let $C = \{1, 2\}$, so that $f^{-1}(C) = 2$, and $f(f^{-1}(C)) = \{1\} \neq C$.

7. Section 4.4 # 5(a)(c), 10, 14(b)(e).

   Solution: 5(a): $\{1, 2, 3\}$.

   5(c): $\{s, z\}$.

   10: First, we prove $f(C \cup D) = f(C) \cup f(D)$:

$$\begin{aligned} y \in f(C \cup D) &\Leftrightarrow \exists x \in C \cup D, f(x) = y \\ &\Leftrightarrow x \in C, f(x) = y, \text{ or } x \in D, f(x) = y \\ &\Leftrightarrow y \in f(C) \text{ or } y \in f(D) \\ &\Leftrightarrow y \in f(C) \cup f(D) \end{aligned}$$

   Next, we prove $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$:

$$\begin{aligned} x \in f^{-1}(E \cup F) &\Leftrightarrow f(x) \in E \cup F \\ &\Leftrightarrow f(x) \in E \text{ or } f(x) \in F \\ &\Leftrightarrow x \in f^{-1}(E) \text{ or } x \in f^{-1}(F) \\ &\Leftrightarrow x \in f^{-1}(E) \cup f^{-1}(F) \end{aligned}$$

   14(b):

$$\begin{aligned} x \in A \setminus f^{-1}(E) &\Rightarrow x \in A, x \notin f^{-1}(E) \\ &\Rightarrow f(x) \in B, f(x) \notin E \\ &\Rightarrow f(x) \in B \setminus E \\ &\Rightarrow x \in f^{-1}(B \setminus E) \end{aligned}$$

   14(e):

$$\begin{aligned} x \in D &\Rightarrow f(x) \in f(D) \\ &\Rightarrow x \in f^{-1}(f(D)) \end{aligned}$$