Homework 5 Solutions

Grading: 6.4 # 6, and problem # 4. Each problem is worth 5 points.

1. Let $P_3(x, y)$ be the set of polynomials of degree at most 3 in the variables x and y.

   (a) Find a basis for P_3 and write its dimension.

   Solution: A basis for $P_3(x, y)$ is given by $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$. (Make sure you use the correct notation here.) So, the dimension is 10.

   (b) Recall that symmetric polynomials are ones where when you switch x and y you get back the polynomial you started with. For example, xy is symmetric, but x^2 is not symmetric. Find a basis for the subspace of P_3 consisting of symmetric polynomials and write its dimension.

   Solution: An arbitrary polynomial looks like:

   $$a + bx + cy + dx^2 + exy + fy^2 + gx^3 + hx^2y + ixy^2 + jy^3.$$

   And if it's symmetric, then this has to equal

   $$a + cx + dy + fx^2 + exy + dy^2 + jx^3 + ix^2y + hxy^2 + gx^3.$$

   Therefore, $b = c$, $d = f$, $g = j$, and $h = i$. Therefore, an arbitrary symmetric polynomial looks like:

   $$a + bx + by + dx^2 + exy + dy^2 + gx^3 + hx^2y + hxy^2 + gy^3.$$

   Rearranging, this is

   $$a + b(x + y) + d(x^2 + y^2) + exy + g(x^3 + y^3) + h(x^2y + xy^2).$$

   Therefore, a basis is given by $\{1, x + y, x^2 + y^2, xy, x^3 + y^3, x^2y + xy^2\}$, so this space is 6-dimensional.

   (c) Give an example of a 2-dimensional subspace of P_3.

   Solution: Any two linearly-independent vectors span a 2-dimensional subspace. So, any two basis vectors span a 2-dimensional subspace, for example, $V = \text{span}\{1, x\}$. (This is a very natural subspace, $P_1(x)$.)

2. Section 6.3: 15, T.5.

   Solution: 15: $c \neq 1$. The easiest way to compute this is to use the fact that the vectors are linearly independent if and only if the determinant of the matrix they form is non-zero. (This only works if the matrix they form is square.)

   T.5: yes, $\{w_1, w_2, w_3\}$ are linearly independent. Here's the proof:

   $$aw_1+bw_2+cw_3 = a(v_1+v_2)+b(v_1+v_3)+c(v_2+v_3) = (a+b)v_1+(a+c)v_2+(b+c)v_3.$$

Setting this quantity equal to zero, by the linear independence of v_1, v_2, v_3, we get

$$\begin{cases} a+b=0 \\ a+c=0 \\ b+c=0 \end{cases},$$

whose solution is $a=b=c=0$.

3. Section 6.4: 6, 30.

   Solution: 6*: Because we know M_2 is 4-dimensional already, we only need to show the four matrices are linearly independent. Because

$$a\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+c & a+d \\ b+d & b+c+d \end{pmatrix},$$

   we get the equations

$$\begin{cases} a+c=0 \\ a+d=0 \\ b+d=0 \\ b+c+d=0 \end{cases},$$

   whose solution is $a=b=c=d=0$, so they are linearly independent.

   30: Again, because we know that $\mathbb{R}^3$ is 3-dimensional, we only need to check linear independence. Putting the vectors into a matrix, and checking when the determinant is non-zero, we get $a \neq 0$, $a \neq \pm 1$.

4. The **length** of a matrix A with real entries is defined to be $\sqrt{\operatorname{tr}(A^tA)}$. (See homework 4 if you've forgotten the definition of tr.) Find the length of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

   Solution*:

$$A^tA = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

   whose trace is 4. Therefore, the length of A is $\sqrt{4} = 2$.