Homework 6 Solutions

1. Consider the set $S = [0, 3) \cup (3, 5)$. What is the interior of S? What are the limit points of S? What is the closure of S?

   Solution: The interior of S is the set $(0, 3) \cup (3, 5)$, the set of limit points is $[0, 5]$, so the closure of S is $[0, 5]$.

2. If A is open and B is closed, prove that $A \backslash B$ is open.

   Solution: Let $x \in A \backslash B$. We want to show that $\exists \epsilon > 0$ so that $B_\epsilon(x) \subseteq A \backslash B$. Since A is open, there exists an ϵ_1 so that $B_{\epsilon_1}(x) \subseteq A$. Now B is closed, so B^c is open. Therefore, since $x \in B^c$, there exists an ϵ_2 so that $B_{\epsilon_2}(x) \subseteq B^c$. Then let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then

$$B_\epsilon(x) = B_{\epsilon_1}(x) \cap B_{\epsilon_2}(x) \subseteq A \cap B^c = A \backslash B.$$

3. Let S be a bounded infinite set and let $x = \sup S$. Prove that either $x \in S$ or x is a limit point of S.

   Solution: Suppose $x \notin S$. Because $x = \sup S$, $\forall \epsilon > 0$, $x - \epsilon$ is not an upper bound of S, so there is some element s of S so that $x - \epsilon < s < x$. But that exactly says that $\forall \epsilon > 0$, $B_\epsilon(x) \cap S \neq \emptyset$, which is the definition of a limit point.

4. Use the definition of compactness to prove that $[1, 3)$ is not compact.

   Solution: For each n, let $U_n = \left(0, 3 - \frac{1}{n}\right)$. Then $[1, 3) \subseteq \bigcup U_n$, but there is no finite subcover.

5. Let $S = \mathbb{R} \times \mathbb{R}$ and let $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ be the Euclidean metric. Define $d^*(x, y) = \min\{1, d(x, y)\}$. Verify that d^* is a metric on S. Draw neighborhoods around the origin of radius 1/2, 1, and 2.

   Solution: We need to check that d^* satisfies the following 4 conditions. All of them follow by the fact that the Euclidean metric satisfies them.

   (a) Non-negativity: $d^*(x, y) \geq 0$.
   If $d(x, y) \leq 1$, then $d^*(x, y) = d(x, y)$ and so non-negativity follows from non-negativity of the Euclidean metric. If $d(x, y) > 1$, then $d^*(x, y) = 1 \geq 0$.

   (b) Positive definite: $d^*(x, y) = 0$ iff $x = y$.
   If $d^*(x, y) = 0$, then $d(x, y) = 0$ which happens iff $x = y$.

   (c) Transitivity: $d^*(x, y) = d^*(y, x)$.
   $d^*(y, x) = \min\{1, d(y, x)\} = \min\{1, d(x, y)\} = d^*(x, y)$.

(d) Triangle inequality: $d^*(x,y) \leq d^*(x,z) + d^*(z,y)$.
In other words, we need to show $\min\{1, d(x,y)\} \leq \min\{1, d(x,z)\} + \min\{1, d(z,y)\}$.
If both $d(x,z) < 1$ and $d(z,y) < 1$, then

$$\min\{1, d(x,y)\} \leq d(x,y) \leq d(x,z) + d(y,z) = \min\{1, d(x,z)\} + \min\{1, d(y,z)\}.$$

Now suppose $d(x,z) \geq 1$. Then

$$\min\{1, d(x,y)\} \leq 1 = \min\{1, d(x,z)\} \leq \min\{1, d(x,z)\} + \min\{1, d(z,y)\} = d^*(x,z) + d^*(z,y).$$

6. If A and B are compact subsets of a metric space (X, d), prove that $A \cup B$ is also compact.

Solution: Let $\{U_\alpha\}$ be an open cover of $A \cup B$. Then $\{U_\alpha\}$ is an open cover of A and an open cover of B, and since each is compact there are finite subcovers $\{U_{i_1}, \dots, U_{i_k}\}$ of A and $\{U_{i_{k+1}}, \dots, U_{i_n}\}$ of B. Then $\{U_{i_1}, \dots U_{i_n}\}$ is a finite subcover of $A \cup B$. Therefore $A \cup B$ is compact.

7. Let x be a point in the metric space (X, d). Prove that the singleton set $\{x\}$ is closed.

Solution: Let $y \in X \backslash \{x\}$. Then $d(x,y) > 0$, so $\epsilon = \frac{d(x,y)}{2} > 0$. Then $B_\epsilon(y) \subseteq X \backslash \{x\}$ so $X \backslash \{x\}$ is open. Therefore $\{x\}$ is closed.