Homework 8 Solutions

Problems graded: # 1, # 2 matrices C and D. 5 points each, total 15 points.

1. Write the matrix
$$A = \begin{pmatrix} 3 & 1 \\ 7 & 2 \end{pmatrix}$$
as the sum of a symmetric matrix and an anti-symmetric matrix. In other words, write $A = B + C$ where $B^t = B$ and $C^t = -C$. Verify that $\langle B, C \rangle = 0$.

   Solution: An arbitrary 2×2 symmetric matrix looks like
$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$
and an arbitrary 2×2 anti-symmetric matrix looks like
$$\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}.$$
So, we set
$$\begin{pmatrix} 3 & 1 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}.$$
This gives us the equations $a = 3$, $b + d = 1$, $b - d = 7$, and $c = 2$. The second and third equations imply that $b = 4$ and $d = -3$. Therefore, $A = B + C$, where
$$B = \begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}.$$
This example hints at the fact that we can use a basis for the symmetric matrices together with a basis for the anti-symmetric matrices to get a basis for the 2×2 matrices. In class we proved that the space orthogonal to the symmetric matrices is the space of anti-symmetric matrices, so we know that $\langle B, C \rangle = 0$. Checking this, we get $\langle B, C \rangle = (3)(0)+(4)(-3)+(4)(3)+(2)(0) = -12+12 = 0$.

2. For each of the following matrices, compute:

   (a) The real eigenvalues of the matrix, and their algebraic multiplicities.

   (b) A basis and dimension for each eigenspace of the matrix.

   (c) The Jordan form of the matrix.

$$A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix},\ B = \begin{pmatrix} 2 & 3 & 3 & 5 \\ 3 & 2 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Solution: The eigenvalues of A are 1 and 6, each with algebraic multiplicity 1. The eigenspace $E_1 = \text{span}\{(1, -3/2)\}$, and the eigenspace $E_6 = \text{span}\{(1, 1)\}$, so both are 1-dimensional, and hence the geometric multiplicity of both eigenvalues is 1. Therefore, A is diagonalizable, so its Jordan form is

$$\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

The eigenvalues of B are 1 with algebraic multiplicity 1, 4 with algebraic multiplicity 1, and 2 with algebraic multiplicity 2. The eigenspaces are $E_1 = \text{span}\{(1, -3, 0, 0)\}$, $E_2 = \text{span}\{(1, 3/2, -3/2, 0)\}$ and $E_4 = \text{span}\{(1, 3/2, 0, 0)\}$. Therefore, the geometric multiplicites of 1, 2 and 4 are all 1. This says that there is one Jordan block for each eigenvalue. Hence the Jordan form is

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The eigenvalues of C are 1 with algebraic mutiplicity 2, and 2 with algebraic multiplicity 1. The eigenspaces are $E_1 = \text{span}\{(1, 0, 0\}$ and $E_2 = \text{span}\{(1, 0, 1)\}$. Therefore, the geometric multiplicites of 1 and 2 are both 1. This says there is one Jordan block for each eigenvalue. Hence the Jordan form is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of D are 1 with algebraic multiplicity 2 and 3 with algebraic multiplicity 1. The eigenspaces are $E_1 = \text{span}\{(1, 0, 0), (0, -2, 1)\}$ and $E_3 = \text{span}\{(1, 0, 1)\}$. Therefore, the geometric multiplicity of 1 is 2, so there are two Jordan blocks with eigenvalue 1, and the geometric multiplicity of 3 is 1, so there is one Jordan block with eigenvalue 3. Therefore, D is diagonalizable, so its Jordan form is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

3. Let A and B be two $n \times n$ matrices that are related by the equation $P^{-1}AP = B$, where P is another $n \times n$ matrix. Prove that $\det(A) = \det(B)$. Hint: Apply the determinant to both sides of the equation $P^{-1}AP = B$ and use determinant properties from section 3.1.

   Solution: The equation tells us that $\det(B) = \det(P^{-1}AP)$. Since $\det(XY) = \det(X)\det(Y)$, we can rewrite this as $\det(B) = \det(P^{-1})\det(A)\det(P)$. Since

$PP^{-1} = I$, we have that $\det(P^{-1}P) = \det(P^{-1})\det(P) = \det(I) = 1$. Since determinants are real numbers which are commutative, we can write

$$\det(B) = \det(P^{-1})\det(A)\det(P) = \det(P^{-1})\det(P)\det(A) = \det(A).$$

4. Let A be an arbitrary 2×2 matrix. Show that the following formula holds for the characteristic polynomial of A:

$$\det(A - \lambda I) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A).$$

Solution: Let A be an arbitrary 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) \end{aligned}$$