# Homework 9 Solutions

Grading: 1(b), 2(B). 5 points each, total 10 points.

1. Suppose that the matrix $A$ has characteristic equation $(\lambda - 2)^3(\lambda + 1)^2$.

   (a) Write all 6 of the possible Jordan forms of $A$.

   **Solution:** The Jordan form of $A$ will have 3 values of $\lambda = 2$ on the diagonal, and 2 values of $\lambda = -1$ on the diagonal. The eigenvalue 2 can be present in either 1, 2, or 3 Jordan blocks, and the eigenvalue $-1$ can be present in either 1 or 2 Jordan blocks. Therefore, there are 6 possibilities, which are:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

   (b) Compute $\det(A)$ and $\operatorname{tr}(A)$.

   **Solution:** Note that all the Jordan forms have the same trace and determinant. The original matrix $A$ has the same trace and determinant as its Jordan form (see homework 8). So, the $\det(A) = (2)(2)(2)(-1)(-1) = 8$, and $\operatorname{tr}(A) = 2 + 2 + 2 - 1 - 1 = 4$. **Note that this means the determinant of a matrix is always the product of its eigenvalues, and the trace is always the sum of its eigenvalues.**

2. Orthogonally diagonalize the following symmetric matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}.$$

   Explicitly write the eigenvalues, a basis for the eigenspaces, and the orthogonal matrix $P$ that you conjugate by to get the diagonalization.

   **Solution:** The eigenvalues of $A$ are 1 and 3. $E_1 = \operatorname{span}\{(1, -1)\}$, and $E_3 = \operatorname{span}\{(1, 1)\}$. Normalize these basis vectors to get the orthogonal conjugating matrix $P$. Therefore,

$$P^t AP = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

The eigenvalues of $B$ are 1 and 5. $E_1 = \text{span}\{(1,0,0),(0,1,1)\}$ and $E_5 = \text{span}\{(0,1,-1)\}$. All these vectors are orthogonal, so normalizing them to get the conjugating matrix $P$, we get

$$P^t AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

3. Let $P_2(x,y)$ be the vector space of polynomials in the variables $x$ and $y$ of degree at most 2. Recall the monomial basis for this space is $\{1, x, y, x^2, xy, y^2\}$.

   (a) Let $D_x$ and $D_y$ be the linear transformations of partial differentiation with respect to $x$ and $y$. Write the matrices of $D_x$ and $D_y$ with respect to the monomial basis.

   **Solution:** $D_x$ is a linear map from $P_2(x,y)$ to $P_1(x,y)$, so that $D_x(a + bx + cy + dx^2 + exy + fy^2) = b + 2dx + ey$. So as a matrix in the standard basis,

$$D_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

   Similarly, the matrix representation of $D_y$ in the standard basis is

$$D_y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

   Note: you could have viewed these maps as having range $P_2(x,y)$ in which case your matrices will have an additional 3 rows of zeroes on the bottom. Since I didn't specify the range, either answer is okay.

   (b) Let $S : P_2(x,y) \to P_2(x,y)$ be the linear transformation which swaps the role of $x$ and $y$ in a polynomial. Write the matrix of $S$ with respect to the monomial basis. Note that $S$ is symmetric.

   **Solution:** The map $S$ has the action $S(a + bx + cy + dx^2 + exy + fy^2) = a + cx + dy + fx^2 + exy + dy^2)$. Therefore, as a matrix in the standard basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that $S$ is symmetric, so we know it's diagonalizable.

(c) Find the eigenvalues and a basis for the eigenspaces of $S$ and orthogonally diagonalize $S$. Note that the eigenspace $E_1$ is the subspace of $P_2(x, y)$ of symmetric polynomials.

The eigenvalues of $S$ are 1 and $-1$. The eigenvectors will be the space of symmetric and anti-symmetric polynomials. In other words,

$$E_1 = \text{span}\{1, x+y, xy, x^2+y^2\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\},$$

and

$$E_{-1} = \text{span}\{x - y, x^2 - y^2\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Normalize all these vectors and write as columns in the conjating matrix $P$. Then the diagonalization of $S$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

So, a basis for the polynomials in which $S$ is diagonal is the set of symmetric/anti-symmetric polynomials, $\{1, x+y, xy, x^2+y^2, x-y, x^2-y^2\}$. This is totally analogous to the symmetric/anti-symmetric basis of the matrices which diagonalizes the transposition map.

4. Let

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Let $L : M_2 \to M_2$ be the linear transformation $L(A) = CA$.

(a) Write the matrix representation of $L$ with respect to the standard basis of $M_2$.

**Solution:** Letting $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary element in $M_2$, we see that $L(v) = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$. Letting the standard basis of $M_2$ be $S$, we see that $(v)_S = (a, b, c, d)$, and $(L(v))_S = (a+2c, b+2d, 3a+4c, 3b+4d)$. The matrix mapping $(v)_S$ to $(L(v))_S$ is

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

(b) Write the matrix representation of $L$ with respect to the symmetric/anti-symmetric basis of $M_2$.

**Solution:** Letting $T$ be the symmetric/anti-symmetric basis, $(v)_T = \left(a, \frac{b+c}{2}, d, \frac{b-c}{2}\right)$, and $(L(v))_T = (a+2c, 2b+3d, 3a+4c, -b-d)$ (if you're unsure how I got this, see the 'notes on coordinate vectors'). The matrix mapping $(v)_T$ to $L(v)_T$ is

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & -1 & -1 & -1 \end{pmatrix}.$$