Calculus 21B

Lecture 4.8

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Organizational details

- All information on this course will be available on the course website

  `http://www.math.ucdavis.edu/~sonya/21B.html`

- Textbook: Thomas' Calculus, Early Transcendentals, 11th ed. (available in the bookstore)
- We will use MyMathLab for graded homework (you will need an access key that either comes with your textbook ("Media Upgrade") or can be purchased separately in the bookstore)
  - Once you have an access key, you need to enroll to this course on `http://www.coursecompass.com/` using the Course Id: `koeppe51267`
- All questions of organizational nature, or how to use MyMathLab, etc. should be directed to our

  Lead Teaching Assistant: Sonya Berg (3206 MSB)

- All questions of mathematical nature should be directed to any of the 3 Teaching Assistants or me.
- Office hours are posted on the course website.

Review: Derivatives as Rates of Change or Linearization

In 21A, we learned to work with derivatives of functions:

$$f'(x_0) = \frac{d}{dx} f(x)\Big|_{x=x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

- Interpretation as the instantaneous rate of change (3.3)
- Interpretation as a linearization (standard linear approximation) (3.10)

$$f(x) \approx L(x) = f(x_0) + f'(x_0) \cdot (x - x_0)$$

- A toolbox of differentiation rules allow us to compute derivatives of any elementary function (given by a closed formula)
  - We know the derivatives of all "primitive" elementary functions (including trigonometric, exponentials, logarithms, etc.)
  - Using rules like the chain rule and the product and the quotient rules, we can combine them.

New Idea of this Course: Reverse this Process

- In many applications, a function F is unknown, but its rate of change is known (or easy to describe).
- We would therefore like to recover a function F from its derivative $F' = f$.

Definition

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

- So we "just reverse" the process we mastered in 21A... boring!?
- Some functions f we immediately recognize as a derivative of a function F
- Differentiation rules (such as linearity of $\frac{d}{dx}$) carry over
- For some functions, we need to work very hard to compute an antiderivative. We will learn several powerful techniques in this course.
- For other functions, such as $f(x) = \frac{\sin x}{x}$, there exists an antiderivative $F(x)$, but it is impossible to write it as a "closed formula" using basic elementary functions.
- Mastering the art of computing antiderivatives, and exploring the boundary between functions for which you can compute antiderivatives and functions for which it is impossible, can be very exciting!

Antiderivatives are unique up to an additive constant

- For a given function f, the antiderivative F is not uniquely determined:

$$\frac{d}{dx}x^5 = 5x^4, \qquad \frac{d}{dx}(x^5 + 91) = 5x^4$$

Are there more antiderivatives?

Lemma

If F is an antiderivative of f on an interval I, then every antiderivative of f on I is of the form

$$F(x) + C \quad \text{(for some constant } C\text{)},$$

(and for any constant C, the function $F(x) + C$ is an antiderivative of f on I).

(This is a consequence of the Mean Value Theorem, which you learned in section 4.2: If F_2 is another antiderivative, then $(F_2 - F)' = f - f = 0$. Then, by Corollary 1 of the Mean Value Theorem, $F_2 - F = C$ for some constant C.)

Indefinite Integrals

- The convention to use uppercase letters $(F, G, H, \ldots)$ for an antiderivative of a function $f, g, h, \ldots$ is not good enough; we want to be able to talk about the antiderivatives of functions that don't have a name.
- Because antiderivatives are unique (only) up to an additive constant C, we introduce a notation for the set of all antiderivatives of a function.
- Notation:

$$\int f(x)\,dx \qquad \text{(indefinite integral)}$$

  Here $\int$ is the integral sign, and $f(x)$ is called the integrand of the integral, and x is the variable of integration.

- This allows us to write:

$$\int 5x^4\,dx = x^5 + C$$

$$\int \left(2x + 3e^{3x}\right) dx = x^2 + e^{3x} + C$$

Antiderivative formulas

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad \text{for } n \neq -1$$

$$\int \sin kx\,dx = -\frac{1}{k}\cos kx + C$$

$$\int \cos kx\,dx = \frac{1}{k}\sin kx + C$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{\sqrt{1-k^2x^2}}\,dx = \frac{1}{k}\sin^{-1} kx + C$$

$$\int \frac{1}{1+k^2x^2}\,dx = \frac{1}{k}\tan^{-1} kx + C$$

(More in the textbook.)

Particular Antiderivatives, and Initial Value Problems

How do we choose a particular antiderivative? Here's a common pattern.

- In addition to the function $f(x)$, we are given the value of the antiderivative $F(x)$ at one point $x = x_0$.
- Example: Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

Here's the mathematical language that we will use to describe problems like this one.

- We can interpret the problem of finding an antiderivative of a function f as a (very simple) differential equation:

  $$\frac{d}{dx}y(x) = f(x)$$

  (The defining characteristics of a differential equation is that the derivative of an unknown function $y(x)$ appears).
- The prescribed value of $y(x)$ at some point $x = x_0$ is called an initial condition:

  $$y(x_0) = y_0$$
- Together, this is called an initial value problem.

Solving initial value problems

Example problem

Find the function $y(x)$ such that

- the slope of the graph at each point (x, y) is $3x^2$,
- the graph must pass through the point $(1, -1)$.

Solution process

1. Solve the differential equation.
   This gives us the general solution of the differential equation.
2. Evaluate the general solution.
   This gives us the particular solution that satisfies the initial condition.

Another initial value problem

A problem in physics (accelerated motion):

- A baloon ascending at the rate of 12 ft/sec is at a height of 80 ft when a package is dropped. How long does it take until the package hits the ground?

We use functions of time here:

- The position $s(t)$ (height over ground)
- The velocity $v(t)$
- The acceleration $a(t)$

We have:

$$v(t) = \frac{d}{dt}s(t), \qquad a(t) = \frac{d}{dt}v(t).$$