Calculus 21B

Lecture 5.1

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Organizational details

- Slides are posted on the course website.
- Lecture on this Friday will be given by my colleague Peter Malkin.
- No office hours on this Friday afternoon; I can make up for this next week if necessary.
- Please enroll in the course on MyMathLab; homework will be assigned starting this evening.
- Homework, continuous instructions:
  - Skim the section of the textbook before class.
  - Read the section of the textbook after class.
  - Graded homework will be continuously assigned on MyMathLab; I will not announce this every time in class. There will be one week of time for every homework assignment. Check the due date in MyMathLab.
- Grading based on:
  - 20% Homework (MyMathLab)
  - 30% Midterms (there will be 2 midterms; I drop the lower grade)
  - 50% Final
- Exam Schedule to be announced.

Two similar examples

Suppose we know the velocity $v(t)$ (given in mph, say) of a truck driving on a highway, without changing direction.
Because the odometer is broken, we would like to compute the distance traveled between times $t = a$ and $t = b$ (given in hours, say).

The Trigonometric Truck

The truck driver, obsessed with trigonometric functions, maintains (on an otherwise empty road) an exact velocity given by the formula

$$v(t) = 40 + 10 \sin t.$$

An Ordinary Truck

The truck driver adapts the velocity according to traffic conditions.
The velocity $v(t)$ is a function of time t, but there is no formula for this function known.
However, we can use the speedometer to determine the velocity $v(t)$ at times $t = t_1, t_2, \ldots, t_n$.

The Trigonometric Truck: as an Antiderivative Problem

- The velocity function $v(t)$ is the derivative of the position function $s(t)$.
- Thus, $s(t)$ is an antiderivative of $v(t)$.
- We can also say that $s(t)$ is a solution of the differential equation

$$\frac{d}{dt}s(t) = v(t).$$

- Let's compute it... from $v(t) = 40 + 10\sin t$

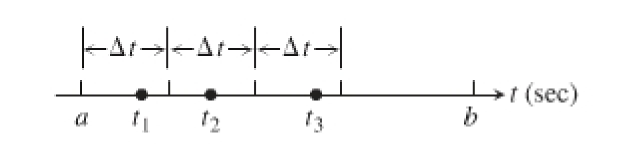
$$s(t) = 40t - 10\cos t + C.$$

  This is the general solution of the diff. equation. Which constant C?
- The distance traveled is the change in position: $D = s(b) - s(a)$.
- So it turns out that the constant C does not matter:

$$\begin{aligned} D = s(b) - s(a) &= \big(40b - 10\cos b + C\big) - \big(40a - 10\cos a + C\big) \\ &= 40(b-a) - 10(\cos b - \cos a). \end{aligned}$$

The Ordinary Truck: Approximating the Distance

- Here the function $v(t)$ is not given by a formula, so we cannot write down an antiderivative!
- Let's subdivide the time interval $[a, b]$ into n subintervals of equal length Δt



and pick a time $t_1, t_2, \ldots, t_n$, one from each interval.
- We read the speedometer at these times. We assume that the velocity will be "more or less constant" during each subinterval.
- So we can compute the distance traveled on each subinterval as

$$\text{Distance} = \text{Velocity} \times \text{Time}.$$

The Ordinary Truck: Approximating the Distance

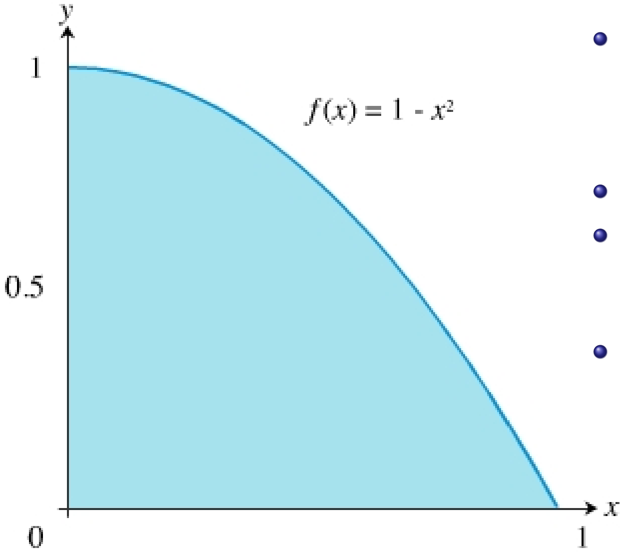
- In total, we get an approximation of the distance

$$D \approx v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t.$$

- We hope that, by making the subintervals short enough, the approximation will be good.

Here's a new type of question

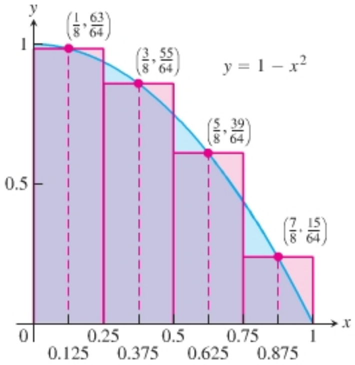
Here's a function $f(x)$.



- The graph of the function in the interval $[0, 1]$ and the two coordinate axes cut out a region.
- What is its area?
- (If this were a triangle or a circle, we would know a formula!)
- Integral calculus (antiderivatives!) will tell us the answer, via something known as the Fundamental Theorem of Calculus

But let's start with a simple idea

We cannot compute the area because of the curved boundary, so let's do something else that is simpler.



Let us approximate the area of the region by summing up areas of rectangular strips

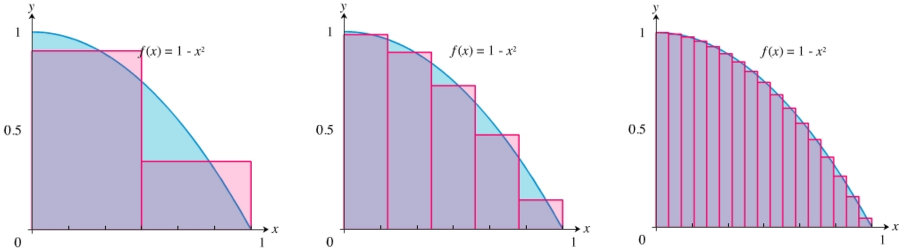
- Divide the interval into some subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $[x_3, x_4]$ (not necessarily of equal length)
- Pick some point c_1 from $[x_0, x_1]$, pick some point c_2 from $[x_1, x_2]$, etc. (not necessarily the midpoint)
- We use a rectangle of height $f(c_1)$ whose base is the interval $[x_0, x_1]$, etc.
- The area of each rectangle is height $\times$ width, so $f(c_1) \cdot (x_1 - x_0)$ etc.

Then (we hope!) $\mathit{Area} \approx \mathit{Area} = f(c_1) \cdot (x_1 - x_0) + f(c_2) \cdot (x_2 - x_1) + f(c_3) \cdot (x_3 - x_2) + f(c_4) \cdot (x_4 - x_3)$

Riemann sums as approximations

- The sum $$\textit{Area} = f(c_1)\cdot(x_1 - x_0) + f(c_2)\cdot(x_2 - x_1) + f(c_3)\cdot(x_3 - x_2) + f(c_4)\cdot(x_4 - x_3)$$ is called a Riemann sum.
- Intuitively clear is this (we will make it precise later):

  If we divide our interval into more (and smaller) subintervals, then the approximation Area ≈ Area should get better.



- The textbook contains numerical examples that illustrate this, and MyMathLab (Figure 5.1 Animation) lets you experiment with the subdivisions and will show you the value of the resulting Riemann sum.

Lower and upper sums

- A problem with these Riemann sums: We don't seem to have control over the error we make in the approximation.
  - The area of some of the rectangles overestimates the true area;
  - the area of other rectangles underestimates the true area.

  In total, for now we can just hope that the approximation is close to the true area of the region, but we don't know whether is above or below, and what the error is.
- Here's a new idea: After choosing the subdivision of the interval, we make a specific choice for all the points c_i in the respective subintervals $[x_{i-1}, x_i]$
  - We can choose all c_i such that $f(c_i)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. The resulting Riemann sum is called an upper sum.
  - Or, we can choose all c_i such that $f(c_i)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. The resulting Riemann sum is called a lower sum.
- Then we have inequalities (estimates):

$$\text{Lower Sum} \leq \text{Any Riemann Sum} \leq \text{Upper Sum}$$

and

$$\text{Lower Sum} \leq \text{True Area of the Region} \leq \text{Upper Sum}$$

Bounding the error

If we have computed both a lower and an upper sum, we have an upper bound for the error that we make by using the approximation instead of the true area. So now the error is under control:

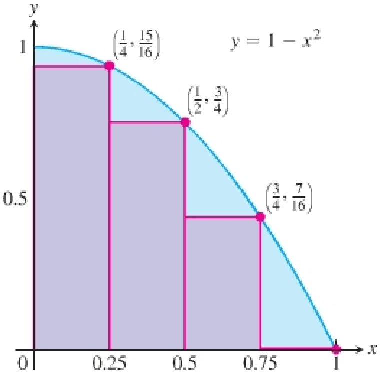
$$\begin{aligned}\text{Error magnitude} &= \left|\text{True Area of the Region} - \text{Riemann Sum}\right| \\ &\leq \text{Upper Sum} - \text{Lower Sum}.\end{aligned}$$

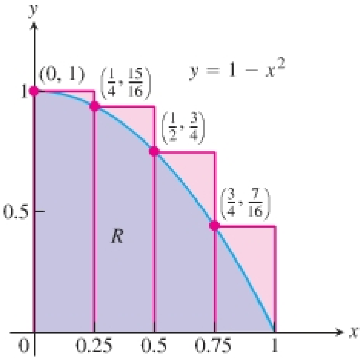
Note that we have an upper bound on the error magnitude without having to know the true area.

- However, lower and upper sums may be hard to compute.
  (We need to find the minimum and the maximum of the function over all the chosen subintervals.)

Lower and upper sums for monotonous functions

- A convenient special case: Monotonous functions.
  - If the function is monotonously decreasing on an interval, then the maximum function value is at the left endpoint, and the minimum function value is at the right endpoint.





  - Likewise, if it is monotonously increasing, then the maximum is at the right, the minimum is at the left.