# Calculus 21B

# Lecture 5.2

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Today's presenter: Dr. Peter Malkin

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# Organizational Reminder

- The first homework is posted on MyMathLab, so enroll now.
- The homework is due Wednesday night.
- Homework counts 20%, so don't miss it.

# Reminder: Riemann sums

**Reminder**

The sum

$$Area = f(c_1)\cdot(x_1 - x_0) + f(c_2)\cdot(x_2 - x_1) + f(c_3)\cdot(x_3 - x_2) + f(c_4)\cdot(x_4 - x_3)$$

is called a Riemann sum.

- These are just 4 terms, imagine what it would look like if we had 100000 terms.
- We really want to be able to talk about Riemann sums with many summands or a varying number of summands.
- Therefore we want a compact notation.
- This is where Sigma Notation comes into play.

# Sigma Notation

- Sigma ($\Sigma\iota\gamma\mu\alpha$) is the name of the (uppercase) Greek letter $\Sigma$, which carries the /s/ sound and is transcribed and transliterated into Latin alphabets as S. It is thought to stand for "sum" (in some language). Note the Greek word for "sum" does not start with a sigma!
- We can write the Riemann sum

$$
\begin{aligned}
Area = {} & f(c_1)\cdot(x_1 - x_0) + f(c_2)\cdot(x_2 - x_1) \\
& + f(c_3)\cdot(x_3 - x_2) + f(c_4)\cdot(x_4 - x_3)
\end{aligned}
$$

  in a compact way using sigma notation as

$$
Area = \sum_{k=1}^{4} f(c_k)(x_k - x_{k-1}).
$$

- That saves a lot of space (imagine how much will be saved for sums with many more terms!)
- Thats's good, so now let's take a look at how to read and use it.

# Sigma Notation

- The idea is that each of the terms
  - $f(c_1) \cdot (x_1 - x_0)$
  - $f(c_2) \cdot (x_2 - x_1)$
  - $f(c_3) \cdot (x_3 - x_2)$
  - $f(c_4) \cdot (x_4 - x_3)$

  takes the form

  $$f(c_k)(x_k - x_{k-1}) \qquad \text{for some } k$$

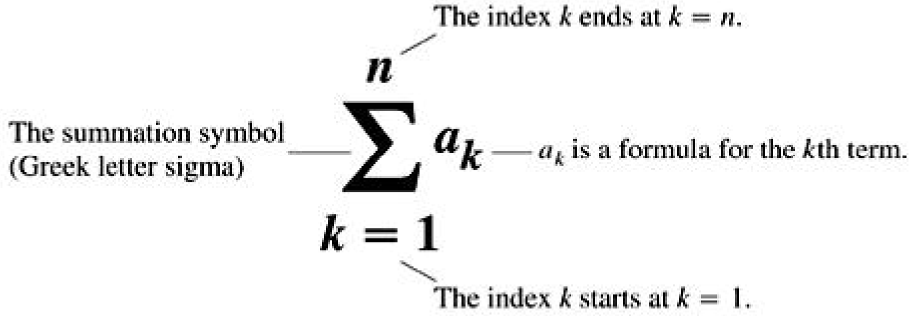
- In fact, if we let $k$ run from 1 to 4, then we get all of these four terms.
- This is what we express by writing

  $$\mathit{Area} = \sum_{k=1}^{4} f(c_k)(x_k - x_{k-1}).$$

- We call $k$ the index of summation.
  (Any variable name can be used. Most of the time we'll be using $i$, $j$, $k$, $l$, and the like.)

# Sigma Notation

Altogether, these are the parts of this notation.



# Some examples

## Examples of sigma notation

| The sum in sigma notation | The sum written out, one term for each value of $k$ | The value of the sum |
|---|---|---|
| $\sum_{k=1}^{5} k$ | $1+2+3+4+5$ | $15$ |
| $\sum_{k=1}^{3} (-1)^k k$ | $(-1)^1(1)+(-1)^2(2)+(-1)^3(3)$ | $-1+2-3=-2$ |
| $\sum_{k=1}^{2} \frac{k}{k+1}$ | $\frac{1}{1+1}+\frac{2}{2+1}$ | $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$ |
| $\sum_{k=4}^{5} \frac{k^2}{k-1}$ | $\frac{4^2}{4-1}+\frac{5^2}{5-1}$ | $\frac{16}{3}+\frac{25}{4}=\frac{139}{12}$ |

Note the trick in the second example to express alternating sums, i.e., sums in which one positive summand is followed by a negative summand, which is followed by a positive summand again, etc.

# More examples

With your help, we will find out what these instances of sigma notation mean.

(a) $\sum_{k=1}^{6} \frac{k-1}{k}$

(b) $\sum_{k=1}^{5} \sin k\pi$

(c) $\sum_{k=1}^{4} (-1)^k \cos k\pi$

With your help, we will be able to express the following sums compactly using sigma notation.

- $1+2+4+8+16+32$
- $1-2+4-8+16-32$
- $-1+4-9+16-25+36$

# About the choice of the limits of summation

- We have expressed

$$-1+4-9+16-25+36=\sum_{k=1}^{6}(-1)^k k^2$$

- Nothing stops us from writing

$$-1+4-9+16-25+36=\sum_{m=3}^{8}(-1)^{m-2}(m-2)^2=\sum_{m=3}^{8}(-1)^m(m-2)^2.$$

- So the same sum can be expressed using different formulas for the summands and different limits of summation. (The two ways are linked by the relation $m=k+2$.)
- In this example, the first way was preferable because the formula for the summands was slightly simpler.
- Sometimes it pays off to shift the limits of summation in a way that will simplify the formula for the summands.

# Algebraic Manipulations

When we have a sum that looks like this:

$$\sum_{k=7}^{9}(k^3+2^k),$$

then it is sometimes useful to do this:

$$\sum_{k=7}^{9}(k^3+2^k) = (7^3+2^7)+(8^3+2^8)+(9^3+2^9)$$

$$= (7^3+8^3+9^3)+(2^7+2^8+2^9) \qquad = \sum_{k=7}^{9}k^3+\sum_{k=7}^{9}2^k.$$

This example makes clear that we can do simple algebraic manipulations with these sums.

# Algebraic Manipulations

- It is important to understand that these algebraic rules are no new insights.
- You know all of these rules about finite sums already; we are only summarizing them again in our new shorthand notation!

**Algebra Rules for Finite Sums**

1. *Sum Rule:* $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule:* $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule:* $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number $c$)
4. *Constant Value Rule:* $\sum_{k=1}^{n} c = n \cdot c$ ($c$ is any constant value.)

# Simple Formulas for Sums

Sometimes there is a simple formula for the result of a long sum.

**The sum of the first $n$ integers**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

**The sum of the first $n$ squares**

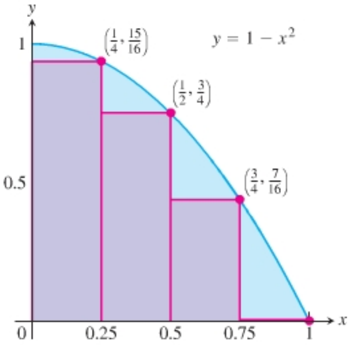
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

**The sum of the first $n$ cubes**

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

The textbook explains how to derive the first summation formula. All of these formulas can be proved using mathematical induction. We just take them as given and use them in the following.

# Back to Riemann Sums




- We want to compute a lower sum for the function $1 - x^2$ on the interval $[0, 1]$, which we considered last time.
- Again?! Yes, but this time we want to know the answer for all possible choices of $n$ (the number of subintervals) at the same time.
- So let us divide $[0, 1]$ into $n$ equal-width subintervals

$$[0, \tfrac{1}{n}],\ [\tfrac{1}{n}, \tfrac{2}{n}],\ \ldots,\ [\tfrac{n-1}{n}, \tfrac{n}{n}]$$

As we said, this is a monotonously decreasing function, so the minimum over each subinterval is attained at the right endpoint. So:

$$\text{Area} \geq \text{Lower Sum} = ??? \text{ (Need your help here!)}$$

(What is the formula for the first rectangular strip, for the second, for the $k$-th?)

# Evaluating the Lower Sum

We have found out

$$\text{Area} \geq \text{Lower Sum} = f(\tfrac{1}{n})(\tfrac{1}{n}) + f(\tfrac{2}{n})(\tfrac{1}{n}) + \cdots + f(\tfrac{k}{n})(\tfrac{1}{n}) + \cdots + f(\tfrac{n}{n})(\tfrac{1}{n})$$

In sigma notation, we have

$$\text{Lower Sum} = \sum_{k=1}^{n} f(\tfrac{k}{n})(\tfrac{1}{n})$$

Using the summation formula for the first $n$ squares, and some of our algebraic manipulation rules, we can compute:

$$\begin{aligned}
\text{Lower Sum} &= \sum_{k=1}^{n} f(\tfrac{k}{n})(\tfrac{1}{n}) = \sum_{k=1}^{n} \left(1 - \left(\tfrac{k}{n}\right)^2\right)(\tfrac{1}{n}) \\
&= \sum_{k=1}^{n} \left(\tfrac{1}{n} - \tfrac{k^2}{n^3}\right) = \sum_{k=1}^{n} \tfrac{1}{n} - \sum_{k=1}^{n} \tfrac{k^2}{n^3} = n \cdot \tfrac{1}{n} - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = 1 - \frac{2n^3 + 3n^2 + n}{6n^3}
\end{aligned}$$

So now we have an exact, general expression for this lower sum, for any $n$!

# Convergence of the Lower and Upper Sums

Here is that expression again:

$$\text{Lower Sum} = 1 - \frac{2n^3 + 3n^2 + n}{6n^3}.$$

In the same way, we can handle the upper sum and obtain

$$\text{Upper Sum} = 1 - \frac{2n^3 + 2n^2 + n}{6n^3}.$$

Based on intuition and experimental evidence (you did try it out on MyMathLab, right?), we claimed that the lower and upper sums (as well as any Riemann sum) should get close to the true answer, when the number $n$ of subintervals is increased.

But now we can show this rigorously for this example (at least for our choice of subintervals of equal length):

$$\lim_{n\to\infty}\left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - \frac{2}{6} = \frac{2}{3}.$$

$$\lim_{n\to\infty}\left(1 - \frac{2n^3 + 2n^2 + n}{6n^3}\right) = 1 - \frac{2}{6} = \frac{2}{3}.$$

# Convergence of the Lower and Upper Sums

- From last time, we know

$$\text{Lower Sum} \leq \text{Any Riemann Sum} \leq \text{Upper Sum}$$

and

$$\text{Lower Sum} \leq \text{True Area of the Region} \leq \text{Upper Sum}$$

- In the limit for $n \longrightarrow \infty$, both the Lower Sum and the Upper Sum converge to $\frac{2}{3}$.
- The value of any Riemann sum and the True Area of the Region are squeezed in between the lower sum and the upper sum.
- Therefore, the True Area of the Region must be $\frac{2}{3}$.
- This method has worked surprisingly well!
- So we now have obtained a complete answer to the question “what is the area of this shape” that is no longer just an approximation.

# Towards a Mathematically Precise Definition of Area

- Looking back, we never had a precise definition of what the area of a shape is, however…
- So we'll be looking for a way to define the notion of the area, based on the convergence of Riemann sums.
- It is mathematically not convincing to base this definition on a partition of the interval into subintervals of equal length.
- Instead, we will be using arbitrary partitions (subdivisions of the interval into subintervals) and their associated Riemann sums.
- Then, it is not sufficient to just have many subintervals! Consider this subdivision, which I will also draw on the blackboard for some values of $n$:

  $$[0, \tfrac{1}{2}],\ [\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2n}],\ [\tfrac{1}{2} + \tfrac{1}{2n}, \tfrac{1}{2} + \tfrac{2}{2n}],\ \ldots,\ [\tfrac{1}{2} + \tfrac{n-1}{2n}, \tfrac{1}{2} + \tfrac{n}{2n}].$$

  If we let $n \to \infty$, we cannot expect the lower and upper sums converge to the same value, because the first subinterval always stays long!
- Next time we will see how to capture the idea of having thin intervals (fine partitions), and how this gives a precise definition of area as a limit of Riemann sums.