Calculus 21B

Lecture 5.3

Matthias Köppe
mkoeppe+21b@math.ucdavis.edu

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Organizational details

Exams:

- All midterms and the final exam will be written, in-class exams
- Midterm #1: Friday May 1, in class
- Midterm #2: Wednesday May 20, in class
- Final: Saturday June 6 at 1 pm.
- No calculators or notes are allowed

The course website now includes an anonymous feedback form

- Please let me know what you like and don't like, and what should be improved.
- Thanks for the feedback already received!

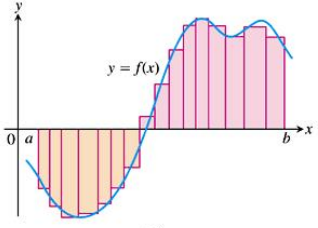
Convergence of Lower and Upper Sums – Reminder

- Reminder: We proved, for the function $f(x) = 1 - x^2$ over the interval $[0, 1]$, the convergence of lower and upper sums, when we use a partition of the interval into subintervals of equal length. Note we had to work hard here, use summation formulas for squares, etc.
- It is mathematically not convincing to base this definition on a partition of the interval into subintervals of equal length.
- Instead, we will be using arbitrary partitions (subdivisions of the interval into subintervals) and their associated Riemann sums.
- Then, it is not sufficient to just have many subintervals! For example, in the partition

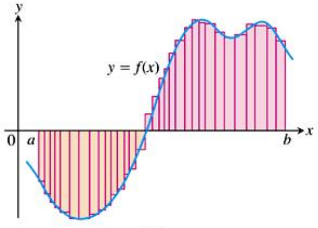
$$[0, \tfrac{1}{2}],\ [\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2n}],\ [\tfrac{1}{2} + \tfrac{1}{2n}, \tfrac{1}{2} + \tfrac{2}{2n}],\ \ldots,\ [\tfrac{1}{2} + \tfrac{n-1}{2n}, \tfrac{1}{2} + \tfrac{n}{2n}],$$

  if we let $n \to \infty$, we cannot expect the lower and upper sums converge to the same value, because the first subinterval always stays long!

Notation for Riemann sums



(a)



(b)

- When subdividing the interval $[a, b]$ into subintervals, we choose $n - 1$ points $x_1, \ldots, x_{n-1}$ with

  $$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

  We call the set

  $$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

  a partition of $[a, b]$
- The norm of the partition P, denoted by $\|P\|$ is the largest length of a subinterval $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$.
- So a small norm indicates a “uniformly fine” subdivision.

Towards the Riemann integral

The Riemann sum of a partition P will be denoted by

$$S_P = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$
$$= \sum_{k=1}^{n} f(c_k)\Delta x_k$$

(Note that it depends also on the points c_k, $k = 1, \ldots, n$.)

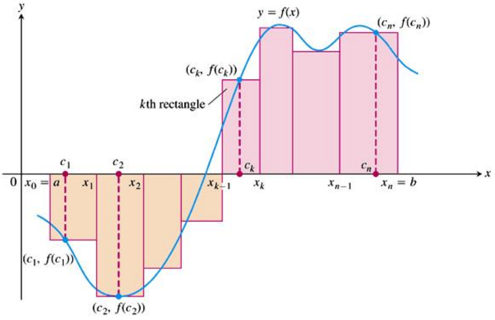


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

Note that if f takes positive and negative values, the Riemann sum approximates the signed area between the graph and the x-axis (this is mathematically more natural, and more useful).

(If we are really interested in the total unsigned area between the graph and the x-axis, we need to break the function into pieces where the sign is constant.)

A remark about convergence

- We are talking about convergence of Riemann sums here, but this is not just convergence of sequences of numbers or the limit of a function of one variable.
- So we cannot just use our notation

$$\lim_{???\to 0} S_P.$$

- Let's start from scratch and use an $\varepsilon - \delta$-style definition of convergence.

The Riemann integral

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$

In this situation:

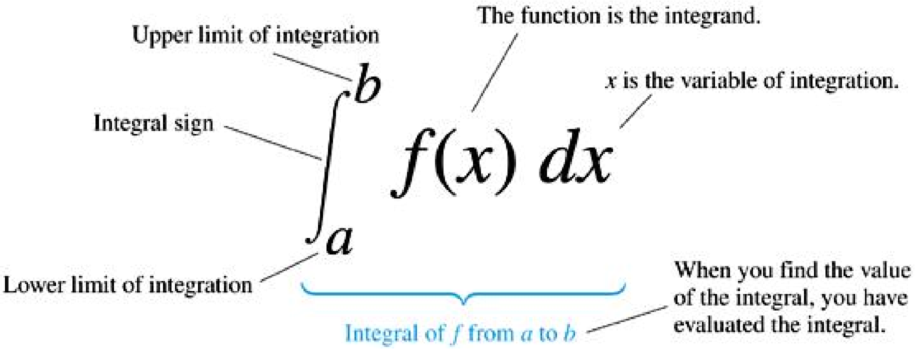
- we use the notation

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = I$$

- we say that the Riemann sums converge to the definite integral I;
- we call the function f (Riemann-) integrable over $[a, b]$.

The definite integral – Notation

If a number *I* with these properties exists, we use this notation for it:



We say: “the integral from a to b of $f(x)$ dee x”

Integrable and non-integrable functions

Theorem (Continuous functions are integrable)

If a function f is continuous on an interval $[a,b]$, then it is integrable over $[a,b]$.

More classes of functions are integrable

- increasing functions
- discontinuous functions with only finitely many jumps

Example of a function that is **not Riemann-integrable** over $[0,1]$

(due to Dirichlet)

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

(Show that upper and lower sums converge to different values.)

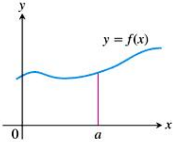
(More sophisticated notions of integration can integrate this function and give the answer 0 “because” there are “way more” irrational numbers than rational numbers. This is the domain of measure theory, a beautiful area of modern mathematics.)

Properties of definite integrals

TABLE 5.3 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

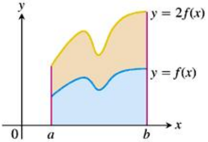
Properties of definite integrals



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

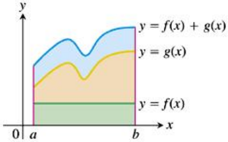
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

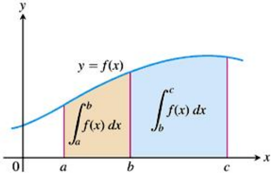
(Shown for $k = 2$.)



(c) *Sum:*

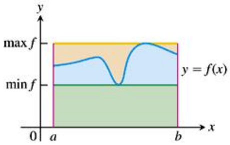
$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)



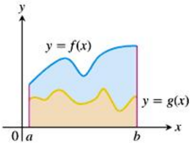
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11