Calculus 21B

Lecture 5.4: The Fundamental Theorem of Calculus

Matthias Köppe
mkoeppe+21b@math.ucdavis.edu

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Organizational details

- This night, 2 am, the first homework is due!
- Please give me feedback on the homework with MyMathLab, using the feedback form.
- Tonight, I will assign new homework using MyMathLab.

Properties of definite integrals

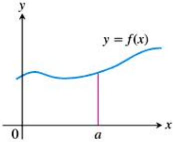
TABLE 5.3 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then	

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

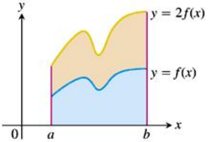
Properties of definite integrals



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

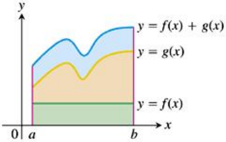
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

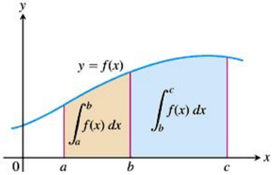
(Shown for $k = 2$.)



(c) *Sum:*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

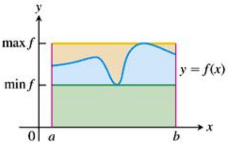
(Areas add)



(d) *Additivity for definite integrals:*

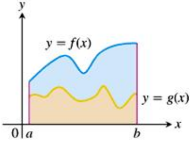
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

FIGURE 5.11



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

The Min-Max Rule

Min-Max Rule

If f has maximum $\max f$ and minimum $\min f$ on $[a, b]$, then

$$\min f \cdot (b-a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b-a).$$

This can be seen as a consequence of the domination rule.

$$f(x) \geq g(x) \text{ for } x \in [a, b] \quad \Longrightarrow \quad \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

So

$$\min f \cdot (b-a) = \int_a^b (\min f)\,dx \leq \int_a^b f(x)\,dx$$

and

$$\max f \cdot (b-a) = \int_a^b (\max f)\,dx \geq \int_a^b f(x)\,dx.$$

Averages and the Mean Value Theorem of Definite Integrals

Average or Mean Value of a Function

We like to define the average of a function f over the interval $[a, b]$ as

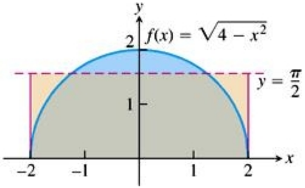
$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

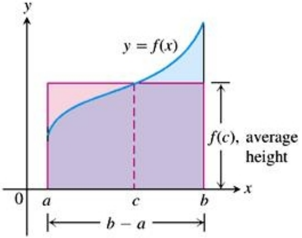
This is a direct analogue of the average of finitely many values.

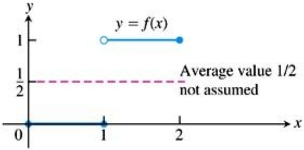
In fact, it is the limit case of the average of finitely many “sampled” values $f(c_1), \ldots, f(c_n)$, when we sample from n subintervals of equal length $\Delta x = (b-a)/n$:

$$\frac{1}{n}\sum_{k=1}^{n} f(c_k) = \frac{1}{n}\sum_{k=1}^{n} f(c_k) = \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k) = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\Delta x$$

Averages and the Mean Value Theorem of Definite Integrals







Theorem (The Mean Value Theorem for Definite Integrals)

If f is continuous on $[a, b]$, then there exists a point $c \in [a, b]$ with

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

(Follows from the Min-Max Rule and the Intermediate Value Theorem for Continuous Functions.)

The textbook shows useful applications of this theorem. We also use it to prove the next theorem.

The Fundamental Theorem of Calculus, Part I

New idea:

- Let f be continuous over the interval $[a, b]$ (and thus integrable over $[a, b]$).
- Then it is also integrable over $[a, x]$ for any $x \in [a, b]$.
- Thus we have a function $F : [a, b] \to \mathbf{R}$ defined by

$$F(x) = \int_a^x f(t)\,dt$$

  (note: no “closed formula”)
- Now $f(x)$ has an interpretation as the instantaneous rate of change of the area $F(x)$

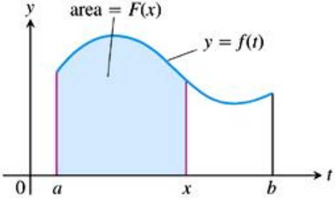
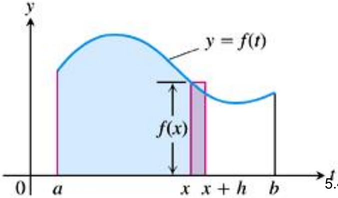


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.



The Fundamental Theorem of Calculus, Part I

Theorem (The Fundamental Theorem of Calculus, Part I)

If f is continuous on $[a, b]$, then

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$, differentiable on (a, b), and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Proof uses the limit definition of a derivative, and the Mean Value Theorem for Definite Integrals.

The Fundamental Theorem of Calculus, Part II

Theorem (The Fundamental Theorem of Calculus, Part II)

If f in continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(Remark: Actually f does not have to be continuous; it suffices that it has an antiderivative on $[a,b]$.)

This gives us a simple procedure to compute definite integrals:

1. Compute an antiderivative.
2. Evaluate it.