Lectures 14 and 15:
Infinite series, convergence tests, and power series

Definitions: sequence of partial sums, infinite sum. Infinite sums which converge, diverge, or diverge to $\pm\infty$.

Examples: geometric series, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, $\sum \sqrt{n+1} - \sqrt{n}$.

Definition: Cauchy criterion for infinite sums.

Proposition: If $\sum a_n$ converges then $a_n \to 0$.

Counterexample to show the converse of the previous proposition does not hold (harmonic series).

Comparison test: Let $\sum a_n$ be a series with $a_n \geq 0\ \forall n$. If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges. If $\sum a_n$ diverges to ∞, and $b_n \leq a_n$ for all n, then $\sum b_n$ diverges to ∞.

Example: $\sum \frac{1}{2^n+1}$.

Definition: absolute convergence.

Proposition: Absolutely convergent series are convergent.

Counterexample to show the converse of the previous proposition does not hold (alternating harmonic series).

Ratio test: Suppose each $a_n > 0$.

1. If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely.
2. If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series diverges.
3. Otherwise, the test gives no information (i.e. there are series with both behaviors).

Root test: Given the series $\sum a_n$, let $\alpha = \limsup |a_n|^{1/n}$.

1. If $\alpha < 1$, then the series converges absolutely.
2. If $\alpha > 1$, then the series diverges.
3. Otherwise $\alpha = 1$ and the test gives no information.

Examples: $\sum \frac{n}{2^n}$, $\sum \frac{n!}{2^n}$, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n}$.

Integral test: Let f be a continuous function defined on $[0, \infty)$ that is both positive and decreasing. Then $\sum f(n)$ converges iff

$$\lim_{n\to\infty} \int_1^n f(x)dx$$

exists as a real number.

Application: the p-series $\sum \frac{1}{n^p}$ converges iff $p > 1$.

Alternating series test: If (a_n) is decreasing and $a_n > 0$ for all n, and if $\lim a_n = 0$, then $\sum(-1)^n a_n$ converges.

Examples: alternating harmonic series, $\sum \frac{\cos(n\pi)}{\sqrt{n}}$.

Definition: power series, radius of convergence.

Ratio criterion: The radius of convergence R of a power series $\sum a_n x^n$ is equal to $\lim \left|\frac{a_n}{a_{n+1}}\right|$, provided the limit exists.

Examples: $\sum x^n$, $\sum \frac{1}{n} x^n$, $\sum \frac{1}{n!} x^n$.