Lecture 1: Introductory set theory, the set of natural numbers and proof by induction.

1. Basic set theory

- Definitions of subset, set equality, union, intersection, disjointedness, empty set, power set, cartesian product.
  Ex: Prove that $C \setminus (A \cup \bar{B}) = \overline{A \cup (\bar{B} \cup \bar{C})}$.
  Note: Any statement "$\forall x \in \emptyset ...$" is automatically true (i.e. a tautology).
- Set notation and indexing families.
  Ex: Let $A_k = [0, 2 - 1/k]$. Find $\bigcup_{k=1}^{\infty} A_k$.
  Ex: Let $B_k = (-1/k, 1/k)$. Find $\bigcap_{k=1}^{\infty} B_k$.
- Definition of function, domain, range, image, pre-image.
  Ex: $f(f^{-1}(D)) \subseteq D$, $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$. What about $\cap$?
- Surjective, injective, and bijective functions. Definition of the inverse, and when it is a function.
  Ex: $f(x) = e^x$, $f(x) = x(x-1)(x+1)$, $f(x) = x^3$.
- Definitions of countable and uncountable.
  Ex: $\mathbb{Z}$ and $\mathbb{Q}$ are countable.
  Ex: $\mathbb{R}$ is uncountable.

2. The natural numbers and proof by induction

- Definition of the set $\mathbb{N}$ of *natural numbers*.
  The set of *natural numbers*, $\mathbb{N}$, is the infinite set $\{0, 1, 2, 3, \dots\}$. (Sometimes this set doesn't contain 0 depending on who you talk to).
  Equivalently, $\mathbb{N}$ is the smallest set containing 0 and closed under the successor function $f(n) = n + 1$.
- Principle of mathematical induction and examples.
  Theorem: Let $P(n)$ be a proposition such that $P(0)$ is true and whenever $P(k)$ is true, $P(k+1)$ is also true. Then $P(n)$ is true for all $n \in \mathbb{N}$.
  In symbols, this theorem is written

$$\forall P, (P(0) \wedge [P(k) \Rightarrow P(k+1)]) \Rightarrow \forall n P(n).$$

  Examples:

$$1 + 2 + \cdots + n = \tfrac{1}{2} n(n+1) \forall n$$
$$n + 1 \leq n^2 \ \forall n \geq 2$$
$$n^2 \leq n! \ \forall n \geq 4$$