Take Home Midterm Exam

MAT 235B / STA 235B

1. You are allowed to use ONLY Durrett's textbook and your lecture notes.

2. No collaboration is allowed

3. Midterms are due Friday, February 12 by 2:10 p.m (in class).

4. Midterm consists of eight problems.

Problem 1

Let $\{\xi_i\}_{i \geq 1}$ be a sequence of i.i.d. random variables, N an independent positive integer valued random variable and

$$X = \xi_1 + \dots + \xi_N.$$

(a) Suppose that ξ_1 and N are integrable random variables. Prove that X is also integrable

(b) Suppose that

$$\mathbb{E}\,\xi_1^{2010} < \infty \quad \text{and} \quad \mathbb{E}\,N^{2010} < \infty$$

Prove that $\mathbb{E}\,X^{2010} < \infty$.

Problem 2

Prove that a class $\mathcal{D}$ of random variables is uniformly integrable iff both of the following conditions hold:

(i) $\mathcal{D}$ is bounded in L^1, i.e.

$$\sup_{X \in \mathcal{D}} \mathbb{E}|X| < \infty.$$

(ii) for every $\varepsilon > 0$ $\exists\, \delta > 0$ such that if A is an event with $\mathbb{P}(A) < \delta$ and $X \in \mathcal{D}$, then

$$\mathbb{E}(|X|; A) := \mathbb{E}(|X| \mathbb{1}_A) < \varepsilon.$$

(Hint: you may wish to use Markov Inequality $\mathbb{P}(|X| > M) \leq \frac{\mathbb{E}|X|}{M}$ in the "if" part)

Problem 3

Suppose that T is a stopping time such that for some integer $M > 0$ and some $\varepsilon > 0$, we have, for every n:

$$P(T \leq n + M \mid \mathcal{F}_n) > \varepsilon \quad \text{a.s.}$$

Prove that $\mathbb{E}(T) < \infty$.

Hint: Prove by induction that

$$P(T > kM) \leq (1-\varepsilon)^k \quad \text{for } k = 1, 2, 3, \ldots$$

Problem 4

Suppose that $X_1, X_2, \ldots$ are i.i.d. Bernoulli random variables with $P(X_i = 1) = p$, $P(X_i = -1) = q$,

$$0 < p = 1 - q < 1, \quad p \neq q.$$

Suppose that a and b are integers with $0 < a < b$. Define

$$S_n := a + X_1 + X_2 + \cdots + X_n,$$

$$T := \inf\{n : S_n = 0 \text{ or } S_n = b\}$$

Explain why T satisfies the conditions of Problem 3 with $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

Calculate $P(S_T = 0)$ and $\mathbb{E}(S_T)$.

Hint: Prove that

$M_n := \left(\frac{q}{p}\right)^{S_n}$ and $N_n = S_n - n(p-q)$ define martingales.