Problem 5

Show that if X is a non-negative supermartingale and T is a stopping time, then

$$\mathbb{E}(X_T ; T < \infty) \leq \mathbb{E}(X_0)$$

(Hint: Use Fatou's lemma)

Deduce that

$$c \, \mathbb{P}(\sup_n X_n \geq c) \leq \mathbb{E}(X_0).$$

Problem 6

(a) Show that if X is a random variable with values in $[-c, c]$ and with $\mathbb{E}(X) = 0$, then for all $t \in \mathbb{R}$,

$$\mathbb{E}\, e^{tX} \leq \cosh(tc) \leq \exp\left(\tfrac{1}{2} t^2 c^2\right)$$

(Hint: use the fact that $f(x) = \exp(tx)$ is convex, so $f(x) \leq \frac{c-x}{2c} f(-c) + \frac{c+x}{2c} f(c)$).

(b) Prove that if $\{M_n\}_{n \geq 0}$ is a martingale with $M_0 \equiv 0$ and such that $|M_n - M_{n-1}| \leq c_n \ \forall n$ for some sequence $\{c_n\}_{n \geq 1}$ of positive constants, then

for, $a > 0$,

$$\mathbb{P}\left(\sup_{0 \leq k \leq n} M_k \geq a\right) \leq \exp\left(\frac{1}{2} \frac{a^2}{\sum_{k=1}^{n} c_k^2}\right).$$

(Hint: use the fact that e^{tM_n} is a submartingale and apply Doob's Inequality).

Problem 7

Let $\{\xi_i\}_{i\geq 1}$ be i.i.d. random variables, $\mathbb{P}(\xi_i = 0) = \mathbb{P}(\xi_i = 2) = \frac{1}{2}$ and let $X_n = \prod_{k=1}^{n} \xi_k$.

(a) Show that $\{X_n\}_{n\geq 1}$ is a martingale

(b) Show that there does not exist an integrable random variable X and a filtration $\{\mathcal{F}_n\}_{n\geq 1}$ such that $X_n = \mathbb{E}(X \mid \mathcal{F}_n)$.

This example shows that not every martingale $\{X_n\}_{n\geq 1}$ can be represented in the form

$$\{\mathbb{E}(X \mid \mathcal{F}_n)\}_{n\geq 1}.$$

Problem 8 (Pólya's urn)

At time 0, an urn contains 1 black ball and 1 white ball.
At each time $n = 1, 2, 3, \ldots$, a ball is chosen at random from the urn and is replaced together with a new ball of the same color.
Thus, just after time n, there are exactly $n+2$ balls in the urn.

Let us denote by $B_n + 1$ the number of black balls in the urn just after time n.

Prove that for $0 < \theta < 1$,

$$N_n^{\theta} := \frac{(n+1)!}{B_n!\,(n-B_n)!}\;\theta^{B_n}(1-\theta)^{n-B_n}$$

is a martingale with respect to a natural filtration which you should specify.