Problem 5

Let X be a Markov chain on a countable set S.

Define $T_y = \inf\{n \geq 1 : X_n = y\}$.

Prove that

$$p^n(x,y) = \sum_{m=1}^{n} \mathbb{P}_x(T_y = m)\, p^{n-m}(y,y).$$

Problem 6

Consider a branching process, i.e. $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \dots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases}$$

with ξ_i^n, $i, n \geq 0$ being i.i.d. non-negative integer-valued random variables.

Assume that $Var(\xi_i^n) > 0$.

Let $G_n(s) = \mathbb{E}\, s^{Z_n}$ be the probability generating function of the size Z_n of the n^{th} generation of a branching process, and H_n be the inverse function of the function G_n, viewed on $[0, 1]$, i.e

$$G_n(H_n(s)) = H_n(G_n(s)) = s \quad \text{for } 0 \leq s \leq 1.$$

Show that $M_n = (H_n(s))^{Z_n}$ defines a martingale w.r.t the natural filtration.

Hint: Recall that

$$G_n(s) = \underbrace{\varphi(\varphi(\dots \varphi(s)) \dots)}_{\leftarrow n \text{ times}}, \text{ where}$$

$$\varphi(s) = \mathbb{E}\, s^{\xi_i^n}.$$

Problem 7

A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.

(a) The largest number X_n shown up to the n^{th} roll.

(b) The number N_n of sixes in n rolls.

(c) At time r, the time C_r since the most recent six.

(d) At time r, the time B_r until the next six.

Problem 8

Let $X_1, X_2, \ldots$ be independent random variables such that

$$X_n = \begin{cases} a_n & \text{with probab. } \frac{1}{2n^2} \\ 0 & \text{with probab. } 1 - \frac{1}{n^2} \\ -a_n & \text{with probab. } \frac{1}{2n^2}, \end{cases}$$

where $a_1 = 2$ and

$$a_n = 4 \sum_{j=1}^{n-1} a_j$$

Show that $Y_n = \sum_{j=1}^{n} X_j$ defines a martingale and that

$$Y_\infty = \lim_{n \to \infty} Y_n$$ exists almost surely.

Show that one can not directly apply the Martingale Convergence Theorem.