Localization at large disorder

Recall:

Last time we introduced

Random Schrödinger Operators

$$H_\lambda(\omega) := -\Delta + \lambda V_\omega \qquad \text{on} \quad \ell^2(\mathbb{Z}^d)$$

where $\lambda \in \mathbb{R}$ is a coupling

and $V_\omega = \{\omega_x\}_{x\in\mathbb{Z}^d}$ is a sequence of i.i.d. random variables

$\Omega = \bigtimes_{x\in\mathbb{Z}^d} [a,b]$ and $d\mathbb{P}(\omega) = \bigtimes_{x\in\mathbb{Z}^d} \rho(\omega_x)d\omega_x$ $\qquad \int_a^b \rho(x)dx = 1$

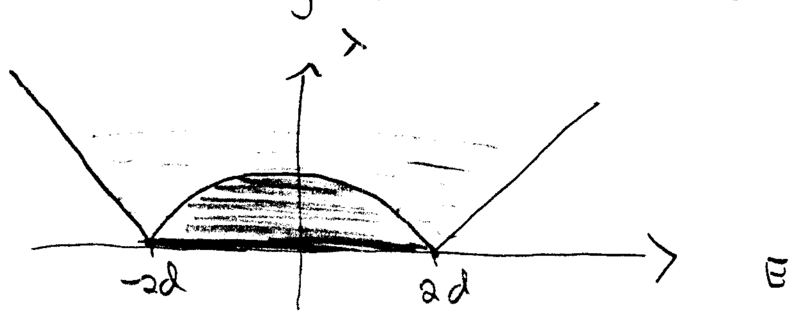
We stated a theorem which said:

Fix $\lambda \in \mathbb{R}$

For $\mathbb{P}$ - a.e. $\omega \in \Omega$

$$\sigma(H_\lambda(\omega)) = \Sigma^\lambda$$
$$\sigma_{pp}(H_\lambda(\omega)) = \Sigma^\lambda_{pp}$$
$$\sigma_{ac}(H_\lambda(\omega)) = \Sigma^\lambda_{ac}$$
$$\sigma_{sc}(H_\lambda(\omega)) = \Sigma^\lambda_{sc}$$

For $d \geq 3$ and ρ the uniform distribution on $[-1,1]$.



phase Transition

$\lambda = 0$

$\sigma(H_0) = [-2d, 2d]$ $\longrightarrow$ increase $\qquad \lambda >> 1$

a.c. $\qquad \sigma(H_\lambda(\omega)) = [-2d-\lambda, 2d+\lambda]$

Some background info.

Suppose $g : [a,b] \to \mathbb{R}$ is a continuous function which is a.s non-constant i.e. $\forall x \in \mathbb{R}$

$$|\{x \in [a,b] : g(x) = x\}| = 0$$

Consider the operator $M_g : L^2([a,b)) \to L^2((a,b))$ given by

$$(M_g f)(x) = g(x) f(x).$$

- This operator is clearly symmetric and self-adjoint.
- Since it is self-adjoint it has a spectral resolution:

Define $\forall t \in \mathbb{R}$ $A_g(t) := \{x \in [a,b] : g(x) \leq t\}$

$$E(t) := \chi_{A_g(t)}$$

It is easy to see that this is the spectral resolution corresponding to M_g.

- For any $f \in L^2([a,b])$

$$\rho_f(t) := \langle E(t) f, f \rangle = \int \chi_{A_g(t)} |f(x)|^2 dx$$

$$= \int_{A_g(t)} |f(x)|^2 dx$$

non-dec. right-cont. function

Thus For any $[c,d] \subset [a,b]$

$$d\beta_f([c,d]) := \beta_f(d) - \beta_f(c)$$

left and right do not matter

$$= \int_{A_g(d)} |f(x)|^2 dx - \int_{A_g(c)} |f(x)|^2 dx$$

$$= \int_{g^{-1}([c,d])} |f(x)|^2 dx$$

$\Rightarrow$ Spectral measure is concentrated on the range of g and if $N \subset \mathbb{R}$ with $|N| = 0$ Then

$$d\beta_f(N) = 0$$

ie. Spectral measures are absolutely continuous.

Thus $\sigma(M_g) = \sigma_{ac}(M_g) = \{ g(x) : x \in [a,b] \}$

Lemma:

Consider

$$H_0 = -\Delta \quad \text{on} \quad \ell^2(\mathbb{Z}^d)$$

Then

$$\sigma(H_0) = \sigma(-\Delta) = \sigma_{ac}(-\Delta) = [-2d, 2d].$$

pf:

Set $d=1$

Denote by

$$F : L^2(0,2\pi) \to \ell^2(\mathbb{Z})$$

$$(Ff)(k) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, e^{-ikx}\, dx$$

Observe that for any $f \in L^2((0,2\pi))$ and $k \in \mathbb{Z}$

$$[H_0(Ff)](k) = (Ff)(k+1) + (Ff)(k-1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) \{ e^{-i(k+1)x} + e^{-i(k-1)x} \} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} 2\cos(x) f(x)\, e^{-ikx}\, dx$$

$$= [F(M_g f)](k)$$

where $g(x) = 2\cos(x)$

$$\Rightarrow \quad H_0 F = F M_g \iff F^{-1} H_0 F = M_g$$

Localization for the Anderson Model at large disorder: The Aizenman - Molchanov Approach:

Ch. 13 Trace ~~Ideals by Barry Simon~~

Theorem: Consider

$$H_\lambda(\omega) := -\Delta + \lambda V_\omega$$

where $V_\omega = \{\omega_n\}_{n \in \mathbb{Z}^d}$ is a multiplication operator by a sequence of i.i.d. random variables. (For simplicity we assume each ω_n is uniformly distributed on $[0,1]$.) There exists $K > 0$ s.t. if $|\lambda| > Kd^2$, then

$\sigma(H_\lambda(\omega)) = \sigma_{pp}(H_\lambda(\omega)) = [-2d, 2d+\lambda]$ a.s.

(i.e. $\sigma_{ac}(H_\lambda(\omega)) = \sigma_{sc}(H_\lambda(\omega)) = \phi$ a.s.)

Pf: Basic idea of the proof:

- Average over the randomness at a single site ($n=0$ e.g.)

- Write

$$H_\lambda(\omega) = -\Delta + \lambda V_\omega = -\Delta + \lambda \sum_{n \in \mathbb{Z}^d} \omega_n \langle \delta_n, \cdot \rangle \delta_n$$

$$= \tilde{H}_\lambda(\omega) + \lambda \omega_0 \langle \delta_0, \cdot \rangle \delta_0$$

independent of the randomness at $n=0$.

Rank 1 perturbation

1st Task: Study Rank one perturbation Theory!

- What quantity to average?
  The Green's function

$$G_\omega(n, 0, E+i0) := \langle \delta_n, (H_\lambda(\omega) - E+i0)^{-1} \delta_0 \rangle$$

<u>Why?</u>

- exponential decay of Green's function
  ⇒ exponential decay of solutions to e.value equation
  ⇒ solutions in $\ell^2(\mathbb{Z}^d)$ i.e. eigenfunctions
  ⇒ point spectrum

rigorous statement

A consequence of Rank One perturbation Theory is the Simon-Wolff Criterion:

"square summability of the Green's function on an interval I ⇒ only pure point spectrum on I".

- Problem: The average Green's function is not square summable:

$$\mathbb{E}\left(|\langle \delta_n, (H_\lambda(\omega))^{-1} - E+i0)^{-1} \delta_0 \rangle|^2\right)$$

This function has singularities

$$\frac{1}{x - E}$$ for all eigenvalues E of $H_\lambda(\omega)$

However, if $a \Leftrightarrow 0<s<1$ then $|G_\omega(n, 0, E+i0)|^s$ is summable! This is the fractional moment method

Rank-1 Perturbation Theory

Let A be a non-negative, self-adjoint operator on a seperable Hilbert space $\mathcal{H}$ (complex).
Let $\phi \in \mathcal{H}$ and define for any $\alpha \in \mathbb{R}$

$$A_\alpha := A + \alpha \langle \phi, \cdot \rangle \phi$$

a family of Rank-1 perturbations of A parametrized by $\alpha \in \mathbb{R}$.

For $z \in \mathbb{C}^+$, define

$$F_\alpha(z) := \langle \phi, (A_\alpha - z)^{-1} \phi \rangle$$

$$\overset{!}{=} \int \frac{d\mu_\alpha(\lambda)}{\lambda - z}$$

see 11.2 for a rigorous discussion.

Borel Transform of μ_α

Some Formalities

$$A - A_\alpha = -\alpha \langle \phi, \cdot \rangle \phi$$

i.e.:

$$(A - z) - (A_\alpha - z) = -\alpha \langle \phi, \cdot \rangle \phi$$

$(A-z)^{-1} \to$ $\leftarrow (A_\alpha - z)^{-1}$

$\square$ $\Rightarrow$ $$(A_\alpha - z)^{-1} - (A - z)^{-1} = -\alpha \langle (A_\alpha - \bar{z})^{-1} \phi, \cdot \rangle (A - z)^{-1} \phi$$

Take "$\langle \phi, \phi \rangle$"

$$F_\alpha(z) - F_0(z) = -\alpha F_\alpha(z) F_0(z)$$

$\square\,\square$ $\Rightarrow$ $$F_\alpha(z) = \frac{F_0(z)}{1 + \alpha F_0(z)}.$$

if $\alpha \neq -\frac{1}{F_0(z)}$

Moreover, applying ϕ to $(\square)$ and using $\boxed{\square\square}$ we have that

$$(A_\alpha - z)^{-1}\phi = (A-z)^{-1}\phi - \alpha F_\alpha(z)\,(A-z)^{-1}\phi$$
$$= [1-\alpha F_\alpha(z)]\,(A-z)^{-1}\phi$$
$$\overset{?}{=} \frac{(A-z)^{-1}\phi}{1+\alpha F_0(z)}$$

Moreover,

$$1-\alpha F_\alpha = 1-\alpha\frac{F_0}{1+\alpha F_0} = \frac{1+\alpha F_0 - \alpha F_0}{1+\alpha F_0}$$
$$= \frac{1}{1+\alpha F_0}$$

$\Rightarrow$

$$(A_\alpha - z)^{-1}\phi = \frac{(A-z)^{-1}\phi}{1+\alpha F_0}$$

$\Rightarrow$

$$\langle \eta, (A_\alpha - z)^{-1}\phi\rangle = \frac{a}{\alpha + b}$$

a crucial Formula!

$$a = \frac{\langle \eta, (A-z)^{-1}\phi\rangle}{F_0(z)} \qquad b = \frac{1}{F_0(z)}$$

both independent of α!

A Theorem or 2

Consider

$$A_\alpha = A + \alpha \langle \phi, \cdot \rangle \phi$$

on a seperable Hilbertspace $\mathcal{H}$ with orthonormal basis $\{\delta_n\}$. For convenience, take $\phi = \delta_0$ i.e. $\|\phi\| = 1$.

We have defined

$$F_\alpha(z) = \langle \delta_0, (A_\alpha - z)^{-1} \delta_0 \rangle$$

consider also the function

$$G(x) := \int \frac{d\mu_0(y)}{(x-y)^2}$$ which takes values in $(0, \infty]$.

Theorem: Let $0 < \alpha < \infty$

Define

$$P_\alpha := \{ x \in \mathbb{R} : F_0(x+i0) = -\tfrac{1}{\alpha} \text{ and } G(x) < \infty \},$$

and

$$L := \{ x \in \mathbb{R} : \operatorname{Im}[F_0(x+i0)] \neq 0 \}.$$

Much More is True !

Then

(i) P_α is the set of eigenvalues for A_α

ie

$$(d\mu_\alpha)_{pp}(x) = \sum_{x_n \in P_\alpha} \frac{1}{\alpha^2 G(x_n)} \delta(x - x_n)$$

(ii) $(d\mu_\alpha)_{ac}$ is supported on L.

Not ind. of α!

<u>Theorem</u>: For each $E \in \mathbb{R}$

$$G(E) = \lim_{\varepsilon \downarrow 0} \sum_n |\langle \delta_n, (A - E - i\varepsilon)^{-1} \delta_0 \rangle|^2 .$$

Lastly Simon-Wolff criterion.

<u>Theorem</u>: Fix $[a,b] \subset \mathbb{R}$.
TFAE.

a) $G(E) < \infty$ for a.e. E in $[a\, b]$ (w.r.t. $|\cdot|$)

b) For a.e. α (w.r.t. $|\cdot|$), A_α has only pure point spectrum in $[a,b]$.

Some of the Basics:

Lemma (Fractional moments)

If $s>1$, then for $\{x_n\}_{n=1}^N$ all positive $N=1,2,\dots,\infty$

$$\sum_{n=1}^N x_n^s \le \left(\sum_{n=1}^N x_n\right)^s .$$

If $0<s<1$, then

$$\sum_{n=1}^N x_n^s \ge \left(\sum_{n=1}^N x_n\right)^s .$$

pf:

If $s>1$, then for any $x_j \in \{x_n\}_{n=1}^N$

$$x_j \le \sum_{n=1}^N x_n$$

$\boxed{\text{as } s-1>0}$

$$\Rightarrow \quad x_j^{s-1} \le \left(\sum_{n=1}^N x_n\right)^{s-1}$$

$\Rightarrow$

$$x_j^s = x_j\, x_j^{s-1} \le x_j \left(\sum_{n=1}^N x_n\right)^{s-1}$$

$$\sum_{j=1}^N x_j^s \le \sum_{j=1}^N x_j \left(\sum_{n=1}^N x_n\right)^{s-1} = \left(\sum_{n=1}^N x_n\right)^s \quad \checkmark$$

if $0<s<1$, then $s-1<0$

$$x_j^{s-1} \ge \left(\sum_{n=1}^N x_n\right)^{s-1}$$

and one may repeat the above proof!

Take $s = 1/4$, then

$$\left(\sum_n |G_\omega(n,0,E+i\varepsilon)|^2 \right)^{1/4} \le \sum_n |G_\omega(n,0,E+i\varepsilon)|^{1/2}$$

Thus if we can prove that

$$\sum_n \mathbb{E}\left(|G_\omega(n,0;E+i\varepsilon)|^{1/2} \right) < \infty \quad \text{a.e. } E \in [a,b]$$

unif. in ε!

Then $\sum_n |G_\omega(n,0,E+i0)|^2 < \infty$ for a.e. (E,ω)!

Lemma: $\exists\, C_0 > 0$ s.t. for all $\alpha, \beta \in \mathbb{C}$

$$\int_0^1 \frac{|x-\alpha|^{1/2}}{|x-\beta|^{1/2}}\, dx \ge C_0 \int_0^1 \frac{1}{|x-\beta|^{1/2}}\, dx$$

Pf:

Note: $|x-\alpha| = \sqrt{(x-\mathrm{Re}(\alpha))^2 + \mathrm{Im}[\alpha]^2} \ge |x - \mathrm{Re}[\alpha]|$

Moreover, if $\alpha \notin [0,1]$, then $|x-\alpha| \ge |x-\alpha^*|$

where α^* is the nearest point to α in $[0,1]$.

Thus we assume $\alpha \in [0,1]$.

We prove the result for $\alpha \in [0, 1/2]$

If $\alpha \in (1/2, 1]$ use the symmetry $x \to 1/2 - x$

(i.e. if $\alpha \in (1/2, 1] \to \tilde{\alpha} := \alpha - 1/2 \in [0, 1/2]$

$y = x - 1/2$

$$\int_0^1 \frac{|x - 1/2 - (\alpha - 1/2)|^{1/2}}{|x-\beta|^{1/2}}\, dx = \int_{-1/2}^{1/2} \frac{|y - \tilde{\alpha}|^{1/2}}{|y - \beta + 1/2|^{1/2}}\, dy$$

Suppose $\alpha \in [0, 1/2]$, then

$$\int_0^1 \frac{|x-\alpha|^{1/2}}{|x-\beta|^{1/2}}\,dx \geq \int_{3/4}^1 \frac{|x-\alpha|^{1/2}}{|x-\beta|^{1/2}}\,dx$$

$$\geq \sqrt{\tfrac{1}{4}} \int_{3/4}^1 \frac{1}{|x-\beta|^{1/2}}\,dx$$

Let

$$h(\beta) := \frac{\displaystyle\int_{3/4}^1 \frac{1}{|x-\beta|^{1/2}}\,dx}{\displaystyle\int_0^1 \frac{1}{|x-\beta|^{1/2}}\,dx}$$

$\alpha \to 1/2$

This is a continuous function on $\mathbb{C}$.
It does not vanish. Thus on every compact subset of $\mathbb{C}$

$$\inf_{\beta \in K} h(\beta) > 0.$$

Moreover,

$$\lim_{\beta \to \infty} h(\beta) = \lim_{\beta \to \infty} \left(\frac{\frac{1}{\sqrt{\beta}} \displaystyle\int_{3/4}^1 \frac{1}{|1-\frac{x}{\beta}|^{1/2}}\,dx}{\frac{1}{\sqrt{\beta}} \displaystyle\int_0^1 \frac{1}{|1-\frac{x}{\beta}|^{1/2}}\,dx} \right)$$

$$= 1/4$$

Thus $\inf_{\beta \in \mathbb{C}} h(\beta) > 0.$ ✓

Lemma:

a) Let $f, g \in \ell^\infty(\mathbb{Z}^d)$ be non-negative and suppose that

$$(1 - aH_0) f \le g$$

for some $0 < a < \frac{1}{2d}$ then

$$f \le (1 - aH_0)^{-1} g$$

b) If $0 < a < \frac{1}{2d}$, then

$$(1 - aH_0)^{-1}(i,j) \le (2da)^{|j-i|} (1 - 2da)^{-1}$$

pf: $\|H_0\| = 2d$ so

$$(1 - aH_0)^{-1} = \sum_{n=0}^{\infty} (aH_0)^n \qquad \text{if } 2ad < 1$$

$$f = (1 - aH_0)^{-1}(1 - aH_0) f$$

$$\le (1 - aH_0)^{-1} g$$

since positivity preserving.

Clearly

$$(1 - aH_0)^{-1}(i,j) = \sum_{n=0}^{\infty} \langle \delta_i, (aH_0)^n \delta_j \rangle$$

$$= \sum_{n=|i-j|}^{\infty} \langle \delta_i, (aH_0)^n \delta_j \rangle \le \sum_{n=|i-j|}^{\infty} (2da)^n$$

proof of Theorem:

Fix $z \in \mathbb{C}^+ = \{ z \in \mathbb{C} : \mathrm{Im}[z] > 0 \}$.
Let

$$g_\omega(n) := G_\omega(n, 0; z)$$

and $K(n) := \mathbb{E}\left(|g_\omega(n)|^{1/2} \right)$

Claim:

$$* \quad K(n) \le \left(C_0 \lambda^{1/2} \right)^{-1} \left[\left(\sum_{|j|=1} K(n+j) \right) + \delta_0(n) \right]$$

pf:
Clearly

$$(H_\lambda(\omega) - z)(H_\lambda(\omega) - z)^{-1} = I$$

$$\Rightarrow \quad \langle \delta_n, (\ )(\ ) \delta_0 \rangle = \delta_0(n)$$

$$\Rightarrow \quad \sum_{|j|=1} g_\omega(n+j) + (\lambda \omega_n - z) g_\omega(n) = \delta_0(n)$$

Lemma
$$\Rightarrow \quad |\lambda \omega_n - z|^{1/2} |g_\omega(n)|^{1/2} \le \delta_0(n) + \sum_{|j|=1} |g_\omega(n+j)|^{1/2}$$

$$\Rightarrow \quad \mathbb{E}\left(|\lambda \omega_n - z|^{1/2} |g_\omega(n)|^{1/2} \right) \le \delta_0(n) + \sum_{|j|=1} K(n+j)$$

Fix $\{\omega_m\}_{m \neq n}$. Then by Rank 1 Theory

$$(*) \quad g_\omega(n) = \frac{c}{\lambda \omega_n + d} \quad \text{with } c \neq 0 \text{ and}$$

c, d functions of $\{\omega_m\}_{m \neq n}$

<u>Thus</u> averaging over ω_n we find

$$\int_0^1 |\lambda \omega - z|^{1/2} \left| \frac{c}{\lambda \omega + d} \right|^{1/2} d\omega$$

$$= |\lambda|^{1/2} \frac{|c|^{1/2}}{|\lambda|^{1/2}} \int_0^1 \frac{|\omega - \frac{z}{\lambda}|^{1/2}}{|\omega + \frac{d}{\lambda}|^{1/2}} d\omega$$

$$\geq |c|^{1/2} C_0 \int_0^1 \frac{1}{|\omega + \frac{d}{\lambda}|^{1/2}} d\omega$$

$$= C_0 |\lambda|^{1/2} \int_0^1 \left| \frac{c}{\lambda \omega + d} \right|^{1/2} d\omega$$

Thus

$$\mathbb{E}\left(|\lambda \omega_n - z|^{1/2} |g_\omega(n)|^{1/2} \right) \geq C_0 |\lambda|^{1/2} K(n)$$

This proves $(*)$!

(*) may be rewritten as $\boxed{d\ ?}$

$$(1 - (c_0 |\lambda|^{1/2})^{-1} H_0) K \le (c_0 |\lambda|^{1/2})^{-1} \delta_0(n)$$

Since $z \in \mathbb{C}^+$, $|g_\omega(n)| \le \frac{1}{|\mathrm{Im}[z]|^{1/2}}$

$\Rightarrow K \in \ell^\infty(\mathbb{Z}^d)$.

Applying the Lemma we have that

if $\frac{1}{c_0 |\lambda|^{1/2}} < \frac{1}{(2d)}$ i.e. $\lambda > \left(\frac{2d}{c_0}\right)^2$

Then

$$K \le (c_0|\lambda|^{1/2})^{-1} (1 - (c_0|\lambda|^{1/2})^{-1} H_0)^{-1} \delta_0$$

$$\Rightarrow K(n) \le (c_0|\lambda|^{1/2})^{-1} (1 - (c_0|\lambda|^{1/2})^{-1} H_0)^{-1}(0,n)$$

$$\le (c_0|\lambda|^{1/2})^{-1} \left[\frac{2d}{c_0|\lambda|^{1/2}}\right]^{|n|} \left(1 - \frac{2d}{c_0|\lambda|^{1/2}}\right)^{-1}$$

Thus

$\sum_n K(n)$ is bounded uniformly in z

$$\Rightarrow \sup_z \mathbb{E}\left(\left(\sum_n |G_\omega(n,0;z)|^2\right)^{1/2}\right)$$ is bounded uniformly in z!