Fluctuations of Linear Eigenvalue Statistics of Random Band Matrices

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Abstract

In this paper, we study the fluctuation of linear eigenvalue statistics of Random Band Matrices defined by $M_n = \frac{1}{\sqrt{b_n}} W_n$, where W_n is a $n \times n$ band Hermitian random matrix of bandwidth b_n, i.e., the diagonal elements and only first b_n off diagonal elements are nonzero. We study the linear eigenvalue statistics $\mathcal{N}(\phi) = \sum_{i=1}^{n} \phi(\lambda_i)$ of such matrices, where λ_i are the eigenvalues of M_n and ϕ is a sufficiently smooth function. We prove that $\sqrt{\frac{b_n}{n}} \left[\mathcal{N}(\phi) - \mathbb{E}\mathcal{N}(\phi)\right] \xrightarrow{d} N(0, V(\phi))$ for $b_n >> \sqrt{n}$, where $V(\phi)$ is given in the Theorem 1.

Keywords: Band random matrix, Central limit theorem, Gaussian distribution, linear eigenvalue statistics, semi circular law, Wigner matrix.

1 Introduction:

Random Matrix Theory was developed from several different sources in the early 20th century. It is used as an important mathematical tool in various fields namely, Mathematics, Physics, wireless communication engineering etc. One of the earliest example of a random matrix appeared in the study of sample covariance estimation was done by John Wishart [**?**]. In the early 1950s, Wigner introduced random matrix ensemble to study the energy spectra of heavy atoms undergoing slow nuclear reactions.

Random matrices are also used to model wireless channels. A random matrix model of CDMA networks can be found in [**?**, **?**].

A special kind of random matrix ensemble is a random band matrix. In 1955, Wigner studied the matrices H of the form $H = K + V$, where K is an $n \times n$ diagonal matrix consisting of $\cdots - 2, -1, 0, 1, 2, \cdots$, and V is an $n \times n$ symmetric sign matrix having non vanishing elements only up to a distance b_n from the main diagonal. Such a matrix H was called as bordered matrix [**?**, **?**].

Another treatment of random band matrix was done by G. Casati et al. [**?**, **?**] in the context of Quantum Chaos. They studied $n \times n$ symmetric random band matrices of bandwidth b_n, where b_n grows with n. In 1992, Molchanov et al. proved the Semicircle Law for random band matrices [**?**]. In 1991, Fyodorov and Mirlin proved that $\frac{b_n^2}{n}$ is a crucial parameter for random band matrices [**?**, **?**]. Numerical simulations show that the local eigenvalue statistics changes from Poisson to GOE or GUE as b_n changes from $b_n << \sqrt{n}$ to $b_n >> \sqrt{n}$. Recently, Li and Soshnikov [**?**] proved the Central Limit Theorem (CLT) for linear statistics of eigenvalues of band random matrices when the bandwidth b_n satisfies $\sqrt{n} << b_n << n$. In this article we write $\alpha_n << \beta_n$ if $\frac{\alpha_n}{\beta_n} \to 0$ as $n \to \infty$, and we write $\alpha_n = O(\beta_n)$ if $\left|\frac{\alpha_n}{\beta_n}\right| \le C$ for all n for some constant $C > 0$.

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In this article, we deal with the CLT for the eigenvalue statistics of band random matrices. We take the approach of M. Shcherbina in [**?**] to establish the CLT for band matrices with bandwidth b_n where $b_n \to \infty$ as $n \to \infty$. We give an alternative proof of Li and Soshnikov [**?**] result on CLT of band matrices when $\sqrt{n} << b_n << n$. We have given some simulation results in Section 5, which ensure that the CLT for band matrices will also hold if $\sqrt{n}/b_n \nrightarrow 0$ and $b_n \to \infty$.

Now we define our model. Let us define the (circular) distance function $d_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as

$$d_n(j,k) := \min\{|j-k|, n-|j-k|\},$$

and the index sets $I_n, I_n^+ \subset \mathbb{N} \times \mathbb{N}$, $I_1 \subset \mathbb{N}$ as

$$I_n := \{(j,k) : d_n(j,k) \le b_n\}, \quad I_n^+ = \{(j,k) : (j,k) \in I_n, j \le k\}, \quad I_1 = \{1 < j \le n : (1,j) \in I_n\} \tag{1}$$

where $\{b_n\}$ is a sequence of positive integers such that $b_n \to \infty$ as $n \to \infty$.

Define a real symmetric random band matrix $M = (m_{jk})_{n \times n}$ of bandwidth b_n as

$$m_{jk} = m_{kj} = \begin{cases} b_n^{-1/2} w_{jk} & \text{if } d_n(j,k) \le b_n \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $\{w_{ii}\}$ and $\{w_{jk}\}_{j \neq k, (j,k) \in I_n^+}$ are two sets of iid real random variables with

$$\mathbb{E}[w_{jk}] = 0, \quad \mathbb{E}[w_{jk}^2] = \begin{cases} 1 & \text{if } j \neq k \\ \sigma^2 & \text{if } j = k. \end{cases} \tag{3}$$

Here $\{w_{jk}\}$ may depend on n, but we suppress it when there is no confusion. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of the random band matrix M. Define the linear eigenvalue statistic of the eigenvalues of M as

$$\mathcal{N}_n(\phi) = \sum_{i=1}^{n} \phi(\lambda_i), \tag{4}$$

and the normalized eigenvalue statistic of the matrix M as

$$\mathcal{M}_n(\phi) = \sqrt{\frac{b_n}{n}} \mathcal{N}_n(\phi), \tag{5}$$

where ϕ is a test function.

2 Main Results:

Theorem 1. *Let M be a real symmetric random band matrix as defined in* (2)*, and b_n be a sequence of integers satisfying $\sqrt{n} << b_n << n$. Assume the following:*

(i) w_{jk} satisfies the Poincaré inequality with constant $m > 0$ not depending on j, k, n i.e., for any continuously differentiable function f,

$$Var(f(w_{jk})) \le \frac{1}{m} \mathbb{E}\left[|f'(w_{jk})|^2\right].$$

(ii) $\mathbb{E}[w_{jk}^4] = \mu_4$ for all $j \neq k$ and $d_n(j,k) \le b_n$.

(iii) $\phi : \mathbb{R} \to \mathbb{R}$ *be a test function in the Sobolev space* H^s *i.e.,* $\|\phi\|_s < \infty$, *where*

$$\begin{aligned} \|\phi\|_s^2 &= \int_{\mathbb{R}} (1+2|t|)^{2s} |\hat{\phi}(t)|^2 \, dt, \\ \hat{\phi}(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\lambda} \phi(\lambda) \, d\lambda, \end{aligned}$$

and $s > 5/2$.

Then the centred normalized eigenvalue statistic $\mathcal{M}^\circ(\phi) = \mathcal{M}_n(\phi) - \mathbb{E}[\mathcal{M}_n(\phi)]$ *converges in distribution to the Gaussian random variable with mean zero and variance given by*

$$\begin{aligned} V(\phi) &= \frac{\kappa_4}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{4-\mu^2}{\sqrt{8-\mu^2}} \phi(\mu) \, d\mu \right)^2 + \frac{\sigma^2}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu\phi(\mu)}{\sqrt{8-\mu^2}} \, d\mu \right)^2 \\ &\quad + \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{(8-x^2)(8-y^2)} F(x,y) \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu_1 \phi(\mu_1)}{(x-\mu_1)\sqrt{8-\mu_1^2}} \frac{\mu_2 \phi(\mu_2)}{(x-\mu_2)^2 \sqrt{8-\mu_2^2}} d\mu_1 d\mu_2 \, dx dy, \end{aligned}$$

where for $x \neq y$

$$F(x,y) = 2 \int_{-\infty}^{\infty} \frac{(s^3 \sin s - s \sin^3 s)}{2(s^2 - \sin^2 s)^2 - (s^3 \sin s + s \sin^3 s) xy + s^2 \sin^2 s (x^2 + y^2)} ds,$$

and κ_4 *is the fourth cumulant of the off-diagonal entries, i.e.,* $\kappa_4 = \mu_4 - 3$.

3 Proof of Theorem 1:

We will follow the approach taken by M. Shcherbina in [**?**] for full (Wigner) matrix. This approach is based on two main ideas. The first ingredient is stated in the following proposition which gives a bound on the variance of linear eigenvalue statistics with a sufficiently smooth test function in term of the variance of the trace of the resolvent of a random matrix. For a proof of this result see [**?**, **?**]. In what follows, we denote $X^\circ = X - \mathbb{E}[X]$ for any random variable X.

Proposition 1. *Let* M *be an* $n \times n$ *real symmetric random matrix and* $\mathcal{N}_n(\phi)$ *be a linear eigenvalue statistic of its eigenvalue as in* (4). *Then for any* $s > 0$ *we have*

$$Var[\mathcal{N}_n(\phi)] \leq C_s \|\phi\|_s^2 \int_0^\infty dy \; e^{-y} y^{2s-1} \int_{-\infty}^{\infty} Var[Tr(G(x+iy))] \, dx,$$

where C_s *is a constant depends only on* s, *and* $G(z) = (M - zI)^{-1}$, *is the resolvent of the matrix* M.

The second ingredient of this approach is to use the martingale difference technique to provide a good bound on $\mathrm{Var}(\gamma_n)$ where γ_n is the trace of the resolvent of a matrix. The following proposition gives that bound.

Proposition 2. *Consider symmetric band matrix* M *defined in* (2) *and assume* (3) *is satisfied. Then for some* $C > 0$ *not depending on* z, n *we have*

$$Var\{\gamma_n\} \leq \frac{Cn}{b_n} \left(y^{-2} + y^{-4} \right) \left(\max\left\{ y, |x| - \frac{2}{y} \right\} \right)^{-2} \tag{6}$$

where $\gamma_n = Tr(M - zI)^{-1} = Tr(G)$ *and* $z = x + iy$, $y > 0$.

We prove this result in the appendix section. Now we outline the proof of Theorem 1

Proof of Theorem 1: By Lévy's continuity theorem, it suffices to show that if

$$Z_n(x) = \mathbb{E}[e_n(x)], \quad e_n(x) = e^{ix\mathcal{M}_n^\circ(\phi)} \tag{7}$$

then for each $x \in \mathbb{R}$

$$\lim_{n\to\infty} Z_n(x) = \exp\left[-\frac{x^2 V(\phi)}{2}\right],$$

where $V(\phi)$ as in Theorem 1. For any test function $\phi \in H^s$, define

$$\phi_\eta = P_\eta * \phi,$$

where P_η is the Poisson kernel given by

$$P_\eta(x) = \frac{\eta}{\pi(x^2+\eta^2)}.$$

We know that ϕ_η approximates ϕ in the H^s norm i.e.,

$$\lim_{\eta\to 0} \|\phi - \phi_\eta\|_s \to 0. \tag{8}$$

For the moment, we denote the characteristic function defined in (7), by $Z_n(\phi)$ (to make its dependence on ϕ clear). Then we have

$$\lim_{n\to\infty} Z_n(\phi) = \lim_{\eta\downarrow 0}\lim_{n\to\infty} (Z_n(\phi) - Z_n(\phi_\eta)) + \lim_{\eta\downarrow 0}\lim_{n\to\infty} Z_n(\phi_\eta).$$

Now using the Proposition 1 and (8), we shall show that

$$\lim_{\eta\downarrow 0}\lim_{n\to\infty} (Z_n(\phi) - Z_n(\phi_\eta)) = 0. \tag{9}$$

and then

$$\lim_{n\to\infty} Z_n(\phi) = \lim_{\eta\downarrow 0}\lim_{n\to\infty} Z_n(\phi_\eta).$$

Hence it suffices to find the limit of

$$Z_{\eta,n} := Z_n(\phi_\eta) = \mathbb{E}\left[e_{\eta,n}(x)\right] \tag{10}$$

with

$$e_{\eta,n}(x) = \exp\left[ix\mathcal{M}_n^\circ(\phi_\eta)\right]$$

as $n \to \infty$ and $\eta \downarrow 0$ uniformly in n. Proofs of (9) and (10) are given in the next two subsections and that will complete the proof of this theorem. $\square$

3.1 Proof of equation (9):

First observe that

$$|Z_n(\phi) - Z_n(\phi_\eta)|^2 \le 2|x|^2 \mathrm{Var}\left[\mathcal{M}_n(\phi) - \mathcal{M}_n(\phi_\eta)\right] \le 2|x|^2 \frac{b_n}{n} \mathrm{Var}\left[\mathcal{N}_n(\phi) - \mathcal{N}_n(\phi_\eta)\right]. \tag{11}$$

Now, in view of Proposition 1, to bound $\mathrm{Var}\left[\mathcal{N}_n(\phi) - \mathcal{N}_n(\phi_\eta)\right]$ we need to estimate

$$\int_{-\infty}^{\infty} \mathrm{Var}\left(\gamma_n(x+iy)\right)\, dx,$$

where $\gamma_n(x+iy) = \mathrm{Tr}(G(x+iy))$ and $G(z) = (M - zI)^{-1}$. We estimate that for $y > 0$

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\max\left\{y, |x| - \frac{2}{y}\right\}\right)^{-2} dx &\leq \int_{||x|-2/y|<y} \frac{1}{y^2}\, dx + \int_{||x|-2/y|\geq y} (x - 2/y)^{-2}\, dx \\
&\leq \frac{10}{y} + 10y
\end{aligned}$$

Using the above estimate and (6), we have

$$\begin{aligned}
\int_0^{\infty} dy\, e^{-y} y^{2s-1} \int_{-\infty}^{\infty} \mathrm{Var}(\gamma_n)\, dx &\leq \frac{C'}{b_n} \int_0^{\infty} e^{-y} y^{2s-1} 4n \left(\frac{1}{y} + y\right)\left(\frac{1}{y^2} + \frac{1}{y^4}\right) dy \\
&= C\frac{n}{b_n} \int_0^{\infty} e^{-y} \left(2y^{2s-3-1} + y^{2s-1-1} + y^{2s-5-1}\right) dy \\
&= C\frac{n}{b_n} \left(\Gamma(2s-3) + \Gamma(2s-1) + \Gamma(2s-5)\right). \qquad (12)
\end{aligned}$$

If we take

$$s = \frac{5}{2} + \epsilon, \quad \epsilon > 0$$

then $\Gamma(2s-3) = \Gamma(2+2\epsilon)$, $\Gamma(2s-1) = \Gamma(4+2\epsilon)$, and $\Gamma(2s-5) = \Gamma(2\epsilon)$. By Proposition 1, and (12), we have

$$\mathrm{Var}\left(\mathcal{N}_n(\phi) - \mathcal{N}_n(\phi_\eta)\right) \leq C(\epsilon)\frac{n}{b_n}\|\phi - \phi_\eta\|_s.$$

Using the above estimate and (11), we have

$$\begin{aligned}
|Z_n(\phi) - Z_n(\phi_\eta)|^2 &\leq 2|x|^2 \frac{b_n}{n} \cdot C(\epsilon)\frac{n}{b_n}\|\phi - \phi_\eta\|_s \\
&= 2C(\epsilon)|x|^2\|\phi - \phi_\eta\|_s \\
&\to 0 \quad \text{as } \eta \to 0.
\end{aligned}$$

The last limit follows from the equation (8). This completes the proof of (9).

3.2 Finding the limit of the characteristic function (10):

We will be using the Lemma 1 and Lemma 2 from appendix in the proof of (10). Let us denote the averaging with respect to $\{w_{1i}; 1 \leq i \leq n\}$ by $\mathbb{E}_1$.

Proof of (10)*:* Using the dominated convergence theorem we have

$$\begin{aligned}
\frac{d}{dx} Z_n(\phi_\eta) &= \frac{d}{dx}\mathbb{E}\left[e_{\eta,n}(x)\right] \\
&= \frac{d}{dx}\mathbb{E}\left[\exp\left(ix\sqrt{\frac{b_n}{n}}\mathcal{N}_n^\circ(\phi_\eta)\right)\right] \\
&= \mathbb{E}\left[i\sqrt{\frac{b_n}{n}}\mathcal{N}_n^\circ(\phi_\eta) e_{\eta,n}(x)\right].
\end{aligned}$$

Since by construction $\phi_\eta = P_\eta * \phi$, we have

$$\mathcal{N}_n^\circ(\phi_\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(\mu)\Im\gamma_n^\circ(z_\mu)\, d\mu, \quad \text{where } z_\mu = \mu + i\eta.$$

Hereinafter, we use the finiteness of $\int_{\mathbb{R}} |\phi(\mu)|\, d\mu$ for $\phi \in H^s, s > \frac{1}{2}$, when changing the order of integration. For notational convenience, from now on we will denote $e_{\eta,n}(x)$ by $e(x)$. Therefore

$$\begin{aligned}
\frac{d}{dx} Z_n(\phi_\eta) &= \mathbb{E}\left[i\sqrt{\frac{b_n}{n}} e(x) \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\mu) \Im \gamma_n^\circ(z_\mu)\, d\mu \right] \\
&= \frac{1}{2\pi} \sqrt{\frac{b_n}{n}} \int_{-\infty}^{\infty} \phi(\mu) \mathbb{E}\left[e(x) \mathrm{Tr}\left(G^\circ(z_\mu) - G^\circ(\bar{z}_\mu) \right) \right]\, d\mu \\
&= \frac{1}{2\pi} \sqrt{\frac{b_n}{n}} \int_{-\infty}^{\infty} \phi(\mu) \left(Y_n(z_\mu, x) - Y_n(\bar{z}_\mu, x) \right)\, d\mu,
\end{aligned}$$

where

$$\begin{aligned}
Y_n(z,x) &= \mathbb{E}\left[e(x) \mathrm{Tr}\left(G^\circ(z) \right) \right] \\
&= \mathbb{E}\left[e^\circ(x) \mathrm{Tr}(G(z)) \right] \\
&= n\mathbb{E}\left[G_{11}(z) e^\circ(x) \right] \\
&= -n\mathbb{E}\left[\left(A^{-1}\right)^\circ e_1(x) \right] - n\mathbb{E}\left[\left(A^{-1}\right)^\circ (e(x) - e_1(x)) \right], \qquad (13)
\end{aligned}$$

$$\begin{aligned}
e_1(x) &= \exp\left[ix\sqrt{\frac{b_n}{n}} \left(\mathcal{N}_{n-1}(\phi_\eta) \right)^\circ \right], \\
\left(\mathcal{N}_{n-1}(\phi_\eta) \right)^\circ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\mu) \Im \left(\gamma_{n-1}(z) \right)^\circ\, d\mu, \\
\gamma_{n-1}(z) &= \mathrm{Tr} G^{(1)}(z), \\
A(z) &= z - \frac{1}{\sqrt{b_n}} w_{11} + \left\langle G^{(1)} m^{(1)}, m^{(1)} \right\rangle, \qquad (14) \\
m^{(1)} &= \frac{1}{\sqrt{b_n}} (w_{12}, w_{13}, \ldots, w_{1n})^T, \qquad (15) \\
G^{(1)}(z) &= \left(G_{ij}^{(1)}(z) \right)_{i,j=2}^{n} = (M^{(1)} - zI)^{-1}, \qquad (16)
\end{aligned}$$

and $M^{(1)}$ is the main bottom $(n-1) \times (n-1)$ minor of M. In the above notation $\langle \cdot, \cdot \rangle$ represents the inner product of two complex vectors, i.e., $\langle x, y \rangle = \bar{y}^T x$ for $x, y \in \mathbb{C}^{n-1}$. The equation (13) follows from the Schur complement lemma, which says that

$$G_{11}(z) = \frac{1}{\frac{1}{\sqrt{b_n}} w_{11} - z - \left\langle G^{(1)} m^{(1)}, m^{(1)} \right\rangle} = -\frac{1}{A(z)}. \qquad (17)$$

Now we rewrite

$$\begin{aligned}
\sqrt{\frac{b_n}{n}} Y_n(z,x) &= -\sqrt{nb_n} \mathbb{E}\left[\left(A^{-1}\right)^\circ e_1(x) \right] - \sqrt{nb_n} \mathbb{E}\left[\left(A^{-1}\right)^\circ (e(x) - e_1(x)) \right] \\
&=: T_1 + T_2. \qquad (18)
\end{aligned}$$

Using Taylor expansion we have

$$A^{-1} = \frac{1}{\mathbb{E}[A]} - \frac{A^\circ}{(\mathbb{E}[A])^2} + \frac{(A^\circ)^2}{(\mathbb{E}[A])^3} - \frac{(A^\circ)^3}{(\mathbb{E}[A])^4} + \frac{(A^\circ)^4}{A(\mathbb{E}[A])^4}. \qquad (19)$$

Therefore, we can estimate

$$T_1 = -\sqrt{nb_n} \mathbb{E}\left[\left(A^{-1}\right)^\circ e_1(x) \right]$$

$$
\begin{aligned}
&= -\sqrt{nb_n}\mathbb{E}\left[\left(A^{-1}\right)e_1^\circ(x)\right] \\
&= -\sqrt{nb_n}\mathbb{E}\left[\left(\frac{1}{\mathbb{E}[A]} - \frac{A^\circ}{(\mathbb{E}[A])^2} + \frac{(A^\circ)^2}{(\mathbb{E}[A])^3} - \frac{(A^\circ)^3}{(\mathbb{E}[A])^4} + \frac{(A^\circ)^4}{A(\mathbb{E}[A])^4}\right)e_1^\circ(x)\right] \\
&= \sqrt{nb_n}\mathbb{E}\left[\left(\frac{A^\circ}{(\mathbb{E}[A])^2} - \frac{(A^\circ)^2}{(\mathbb{E}[A])^3}\right)e_1^\circ(x)\right] + \sqrt{nb_n}\mathbb{E}\left[\left(\frac{(A^\circ)^3}{(\mathbb{E}[A])^4} - \frac{(A^\circ)^4}{A(\mathbb{E}[A])^4}\right)e_1^\circ(x)\right]. \qquad (20)
\end{aligned}
$$

Now we shall estimate each term individually. First of all, since M is a real symmetric matrix we have

$$
\|G(z)\| \leq \frac{1}{|\Im z|}, \tag{21}
$$

and, in particular, $1/|A| \leq 1/|\Im z|$. It can also be checked that $1/|\mathbb{E}[A]| \leq 1/|\Im z|$. Hereinafter $\|X\|$ is the spectral norm of a matrix X. Using the above equation (21) and the estimates (37), (39), we have

$$
\left|\sqrt{nb_n}\mathbb{E}\left[\frac{(A^\circ)^4}{A(\mathbb{E}[A])^4}e_1^\circ(x)\right]\right| \leq \frac{\sqrt{nb_n}}{|\Im z|^5}\mathbb{E}\left[|(A^\circ)^4|\right] = \frac{\sqrt{nb_n}}{|\Im z|^5}O(b_n^{-2}) = O\left(\sqrt{\frac{n}{b_n^3}}\right) \to 0,
$$

$$
\left|\sqrt{nb_n}\mathbb{E}\left[\frac{(A^\circ)^3}{(\mathbb{E}[A])^4}e_1^\circ(x)\right]\right| \leq \frac{\sqrt{nb_n}}{|\Im z|^4}\mathbb{E}\left[|(A^\circ)^3|\right] = \frac{\sqrt{nb_n}}{|\Im z|^4}O(b_n^{-3/2}) = O\left(\sqrt{\frac{n}{b_n^2}}\right) \to 0,
$$

$$
\begin{aligned}
\left|\sqrt{nb_n}\mathbb{E}\left[\frac{(A^\circ)^2}{(\mathbb{E}[A])^3}e_1^\circ(x)\right]\right| &\leq \frac{\sqrt{nb_n}}{|\Im z|^3}\left|\mathbb{E}\left[e_1^\circ(x)\mathbb{E}_1\left[(A^\circ)^2\right]\right]\right| \\
&\leq C\sqrt{\frac{n}{b_n}}\left|\mathbb{E}\left[e_1^\circ(x)\left(b_n\mathbb{E}_1(A^\circ)^2\right)\right]\right| \\
&\leq C\sqrt{\frac{n}{b_n}}\left[\mathrm{Var}(e_1^\circ(x))\right]^{1/2}\left[\mathrm{Var}\left(b_n\mathbb{E}_1(A^\circ)^2\right)\right]^{1/2} \\
&\leq C\sqrt{\frac{n}{b_n}}O(b_n^{-1/2}) \\
&= O\left(\sqrt{\frac{n}{b_n^2}}\right) \to 0, \text{ as } n \to \infty,
\end{aligned}
$$

Therefore, we have

$$
T_1 = \frac{\sqrt{nb_n}}{(\mathbb{E}[A])^2}\mathbb{E}\left[A^\circ e_1^\circ(x)\right] + O\left(\sqrt{\frac{n}{b_n^2}}\right) = \frac{\sqrt{nb_n}}{(\mathbb{E}[A])^2}\mathbb{E}\left[e_1^\circ(x)\mathbb{E}_1(A^\circ)\right] + O\left(\sqrt{\frac{n}{b_n^2}}\right).
$$

Now

$$
A^\circ = -\frac{1}{\sqrt{b_n}}w_{11} + \frac{1}{b_n}\sum_{\substack{i\neq j\\ i,j\in I_1}} G_{ij}^{(1)}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1}\left(G_{ii}^{(1)}w_{1i}^2 - \mathbb{E}[G_{ii}^{(1)}]\right),
$$

where $I_1 = \{1 < j \leq n : (1,j) \in I_n\}$. Therefore,

$$
\mathbb{E}_1[A^\circ(z)] = \frac{1}{b_n}\sum_{i\in I_1}\left(G_{ii}^{(1)} - \mathbb{E}[G_{ii}^{(1)}]\right)
$$

and hence

$$
T_1 = \frac{\sqrt{nb_n}}{(\mathbb{E}[A])^2}\mathbb{E}\left[e_1^\circ(x)\mathbb{E}_1(A^\circ)\right] + O\left(\sqrt{\frac{n}{b_n^2}}\right)
$$

$$
\begin{aligned}
&= \frac{\sqrt{nb_n}}{(\mathbb{E}[A])^2}\mathbb{E}\left[e_1^\circ(x)\frac{1}{b_n}\sum_{i\in I_1}(G_{ii}^{(1)}-\mathbb{E}[G_{ii}^{(1)}])\right]+O\left(\sqrt{\frac{n}{b_n^2}}\right)\\
&= \frac{\sqrt{nb_n}}{(\mathbb{E}[A])^2}\;2\mathbb{E}\left[(G_{22}^{(1)})^\circ e_1^\circ(x)\right]+O\left(\sqrt{\frac{n}{b_n^2}}\right)\\
&= \frac{\sqrt{nb_n}}{(\mathbb{E}[A])^2}\;\frac{2}{n}\;\mathbb{E}[\gamma_{n-1}^\circ e_1^\circ(x)]+O\left(\sqrt{\frac{n}{b_n^2}}\right)\\
&= \sqrt{\frac{b_n}{n}}\frac{2}{(\mathbb{E}[A])^2}\mathbb{E}\left[\gamma_{n-1}^\circ e_1(x)\right]+O\left(\sqrt{\frac{n}{b_n^2}}\right).
\end{aligned} \tag{22}
$$

Hereinafter, all bounds (implicitly) depending on z hold uniformly on the set $\{\mu+i\eta:\mu\in\mathbb{R}\}$ for any given $\eta>0$. Now

$$
\begin{aligned}
\left|\mathbb{E}\left[\gamma_{n-1}^\circ e_1(x)\right]-\mathbb{E}\left[\gamma_n^\circ e(x)\right]\right| &= \left|\mathbb{E}\left[\gamma_{n-1}^\circ e_1(x)\right]-\mathbb{E}\left[\gamma_n^\circ e_1(x)\right]+\mathbb{E}\left[\gamma_n^\circ e_1(x)\right]-\mathbb{E}\left[\gamma_n^\circ e(x)\right]\right|\\
&\le \left(\mathbb{E}\left[\left|\gamma_{n-1}^\circ-\gamma_n^\circ\right|^4\right]\right)^{1/4}+\left|\mathbb{E}\left[\gamma_n^\circ(e_1(x)-e(x))\right]\right|\\
&= O(b_n^{-1/2})+\left|\mathbb{E}\left[\gamma_n^\circ(e_1(x)-e(x))\right]\right|.
\end{aligned}
$$

The last equality follows from (40). We estimate

$$
\begin{aligned}
e(x)-e_1(x) &= \exp\left[ix\sqrt{\frac{b_n}{n}}\mathcal{N}_n^\circ(\phi_\eta)\right]-\exp\left[ix\sqrt{\frac{b_n}{n}}\mathcal{N}_{n-1}^\circ(\phi_\eta)\right]\\
&= \left(\exp\left[ix\sqrt{\frac{b_n}{n}}\mathcal{N}_n^\circ(\phi_\eta)-ix\sqrt{\frac{b_n}{n}}\mathcal{N}_{n-1}^\circ(\phi_\eta)\right]-1\right)e_1(x)\\
&= ix\sqrt{\frac{b_n}{n}}\left(\mathcal{N}_n^\circ(\phi_\eta)-\mathcal{N}_{n-1}^\circ(\phi_\eta)\right)e_1(x)+\frac{b_n}{n}O\left(x^2\left(\mathcal{N}_n^\circ(\phi_\eta)-\mathcal{N}_{n-1}^\circ(\phi_\eta)\right)^2e_1(x)\right)\\
&= \frac{ix}{\pi}\sqrt{\frac{b_n}{n}}\int_{-\infty}^{\infty}\left[\phi(\mu)\Im\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)e_1(x)+\sqrt{\frac{b_n}{n}}\phi(\mu)O(\gamma_n^\circ-\gamma_{n-1}^\circ)^2\right]\,d\mu.
\end{aligned} \tag{23}
$$

Therefore

$$
\mathbb{E}\left[\gamma_n^\circ(e(x)-e_1(x))\right]=\mathbb{E}\left[\frac{ix}{\pi}\sqrt{\frac{b_n}{n}}\int_{-\infty}^{\infty}\phi(\mu)\left[\Im\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)e_1(x)\gamma_n^\circ+\sqrt{\frac{b_n}{n}}\gamma_n^\circ O\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)^2\right]\,d\mu\right].
$$

Using estimates (6) and (40), we have

$$
\left|\mathbb{E}\left[\Im\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)e_1(x)\gamma_n^\circ\right]\right|\le\left(\mathbb{E}[|\gamma_n^\circ|^2]\right)^{1/2}\left(\mathbb{E}\left[\left|e_1(x)\Im\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)\right|^2\right]\right)^{1/2}=O\left(\sqrt{\frac{n}{b_n}}\sqrt{\frac{1}{b_n}}\right).
$$

Similarly,

$$
\mathbb{E}\left[\gamma_n^\circ O\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)^2\right]\quad=\quad O\left(\sqrt{\frac{n}{b_n}}\frac{1}{b_n}\right).
$$

Therefore,

$$
\left|\mathbb{E}\left[\gamma_{n-1}^\circ e_1(x)\right]-\mathbb{E}\left[\gamma_n^\circ e(x)\right]\right|=O\left(\frac{1}{\sqrt{b_n}}\right).
$$

From the equation (22) and the above estimates we have

$$
T_1\quad=\quad\sqrt{\frac{b_n}{n}}\frac{2}{(\mathbb{E}[A])^2}\mathbb{E}\left[\gamma_{n-1}^\circ e_1(x)\right]+O\left(\frac{\sqrt{n}}{b_n}\right)
$$

$$
\begin{aligned}
&= \sqrt{\frac{b_n}{n}}\frac{2}{(\mathbb{E}[A])^2}\left[\mathbb{E}[\gamma_n^\circ e(x)] + O\left(\frac{1}{\sqrt{b_n}}\right)\right] + O\left(\frac{\sqrt{n}}{b_n}\right) \\
&= \sqrt{\frac{b_n}{n}}\frac{2}{(\mathbb{E}[A])^2}Y_n(z,x) + O\left(\frac{\sqrt{n}}{b_n}\right). \qquad (24)
\end{aligned}
$$

Now consider T_2. Using (23) and (30) we have

$$
\begin{aligned}
T_2 &= -\sqrt{nb_n}\mathbb{E}\left[(A^{-1})^\circ(e(x)-e_1(x))\right] \\
&= -\frac{ixb_n}{\pi}\mathbb{E}\left[(A^{-1})^\circ\int_{-\infty}^{\infty}\phi(\mu)\Im\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)e_1(x)\,d\mu\right] \\
&\quad -\frac{1}{\pi}\sqrt{\frac{b_n^3}{n}}\mathbb{E}\left[(A^{-1})^\circ\int_{-\infty}^{\infty}\phi(\mu)O(\gamma_n^\circ-\gamma_{n-1}^\circ)^2\,d\mu\right] \\
&= -\frac{ixb_n}{\pi}\int_{-\infty}^{\infty}\phi(\mu)\mathbb{E}\left[e_1(x)(A^{-1})^\circ\Im(\gamma_n^\circ-\gamma_{n-1}^\circ)\right]\,d\mu + \sqrt{\frac{b_n^3}{n}}O\left(\frac{1}{b_n}\right) \\
&= -\frac{ixb_n}{\pi}\int_{-\infty}^{\infty}\phi(\mu)\mathbb{E}\left[e_1(x)(A^{-1})^\circ\Im\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)\right]\,d\mu + O\left(\sqrt{\frac{b_n}{n}}\right) \\
&= -\frac{ixb_n}{\pi}\int_{-\infty}^{\infty}\phi(\mu)\mathbb{E}\left[e_1(x)(A^{-1})^\circ\Im\left(\gamma_n-\gamma_{n-1}\right)^\circ\right]\,d\mu + O\left(\sqrt{\frac{b_n}{n}}\right) \\
&= \frac{ixb_n}{\pi}\int_{-\infty}^{\infty}\phi(\mu)\mathbb{E}\left[e_1(x)(A^{-1})^\circ(z)\Im\left(\frac{1+B(z_\mu)}{A(z_\mu)}\right)^\circ\right]\,d\mu + O\left(\sqrt{\frac{b_n}{n}}\right) \\
&= T_{21} - T_{22} + O\left(\sqrt{\frac{b_n}{n}}\right),
\end{aligned}
$$

where $B(z) = \left\langle G^{(1)}G^{(1)}m^{(1)}, m^{(1)}\right\rangle$ and

$$
\begin{aligned}
T_{21} &= \frac{xb_n}{2\pi}\int_{-\infty}^{\infty}\phi(\mu)\mathbb{E}\left[e_1(x)(A^{-1})^\circ(z)\left(\frac{1+B(z_\mu)}{A(z_\mu)}\right)^\circ\right]d\mu, \\
T_{22} &= \frac{xb_n}{2\pi}\int_{-\infty}^{\infty}\phi(\mu)\mathbb{E}\left[e_1(x)(A^{-1})^\circ(z)\left(\frac{\overline{1+B(z_\mu)}}{\overline{A(z_\mu)}}\right)^\circ\right]d\mu.
\end{aligned}
$$

Using $\Im\left\langle G^{(1)}m^{(1)}, m^{(1)}\right\rangle = \Im z\left\langle G^{(1)*}G^{(1)}m^{(1)}, m^{(1)}\right\rangle$, it can be easily verified that

$$
\left|\frac{B(z)}{A(z)}\right| \le \frac{1}{|\Im z|},\ \frac{1}{|\mathbb{E}[A(z)]|} \le \frac{1}{|\Im z|},\ \text{and } |\mathbb{E}[B(z)]| \le \frac{2}{|\Im z|^2}. \qquad (25)
$$

Applying $A^{-1} = \frac{1}{\mathbb{E}[A]} - \frac{A^\circ}{(\mathbb{E}[A])^2} + \frac{(A^\circ)^2}{A(\mathbb{E}[A])^2}$ to $A^{-1}(z),\ A^{-1}(z_\mu)$ and using (37), we get

$$
\begin{aligned}
&b_n\mathbb{E}\left[e_1(x)(A^{-1})^\circ(z)\left(\frac{1+B(z_\mu)}{A(z_\mu)}\right)^\circ\right] \\
= \;&b_n\mathbb{E}\left[e_1(x)\left\{\frac{A^\circ(z)}{\mathbb{E}^2[A(z)]}\left(-\frac{B^\circ(z_\mu)}{\mathbb{E}[A(z_\mu)]} + \frac{(1+B(z_\mu))A^\circ(z_\mu)}{\mathbb{E}^2[A(z_\mu)]} - \frac{\mathbb{E}[B(z_\mu)A^\circ(z_\mu)]}{\mathbb{E}^2[A(z_\mu)]}\right)\right\}\right] + O(b_n^{-1/2}) \\
= \;&b_n\mathbb{E}\left[e_1(x)\left\{\frac{A^\circ(z)}{\mathbb{E}^2[A(z)]}\left(-\frac{B^\circ(z_\mu)}{\mathbb{E}[A(z_\mu)]} + \frac{(1+\mathbb{E}[B(z_\mu)])A^\circ(z_\mu)}{\mathbb{E}^2[A(z_\mu)]} + \frac{(B^\circ(z_\mu)A^\circ(z_\mu))^\circ}{\mathbb{E}^2[A(z_\mu)]}\right)\right\}\right] + O(b_n^{-1/2}) \\
= \;&\frac{(1+\mathbb{E}[B(z_\mu)])}{\mathbb{E}^2[A(z)]\mathbb{E}^2[A(z_\mu)]}\mathbb{E}\left[e_1(x)b_nA^\circ(z)A^\circ(z_\mu)\right] - \frac{\mathbb{E}\left[e_1(x)b_nA^\circ(z)B^\circ(z_\mu)\right]}{\mathbb{E}^2[A(z)]\mathbb{E}[A(z_\mu)]} + O(b_n^{-1/2})
\end{aligned}
$$

$$= \frac{(1+\mathbb{E}[B(z_\mu)])}{\mathbb{E}^2[A(z)]\mathbb{E}^2[A(z_\mu)]}\mathbb{E}\left[e_1(x)\mathbb{E}_1\left(b_nA^\circ(z)A^\circ(z_\mu)\right)\right] - \frac{\mathbb{E}\left[e_1(x)\mathbb{E}_1\left[b_nA^\circ(z)B^\circ(z_\mu)\right]\right]}{\mathbb{E}^2[A(z)]\mathbb{E}[A(z_\mu)]} + O(b_n^{-1/2}).$$

Using (39), from the last expression we get

$$\begin{aligned} & b_n\mathbb{E}\left[e_1(x)(A^{-1})^\circ(z)\left(\frac{1+B(z_\mu)}{A(z_\mu)}\right)^\circ\right] \\ = & \frac{(1+\mathbb{E}B(z_\mu))}{\mathbb{E}^2[A(z)]\mathbb{E}^2[A(z_\mu)]}\mathbb{E}[e_1(x)]\mathbb{E}\left[b_nA^\circ(z)A^\circ(z_\mu)\right] - \frac{\mathbb{E}[e_1(x)]\mathbb{E}\left[b_nA^\circ(z)B^\circ(z_\mu)\right]}{\mathbb{E}^2[A(z)]\mathbb{E}[A(z_\mu)]} + O(b_n^{-1/2}). \end{aligned} \tag{26}$$

Define

$$D_n(z,z_\mu) = \frac{(1+\mathbb{E}[B(z_\mu)])\mathbb{E}[b_n\mathbb{E}_1\{A^\circ(z)A^\circ(z_\mu)\}]}{\mathbb{E}^2[A(z)]\mathbb{E}^2[A(z_\mu)]} - \frac{\mathbb{E}[b_n\mathbb{E}_1\{A^\circ(z)B^\circ(z_\mu)\}]}{\mathbb{E}^2[A(z)]\mathbb{E}[A(z_\mu)]}.$$

Also, using (23) and (40), we have

$$\begin{aligned} \mathbb{E}[e(x)] - \mathbb{E}[e_1(x)] &= \mathbb{E}\left[\frac{ix}{\pi}\sqrt{\frac{b_n}{n}}\int_{-\infty}^{\infty}\phi(\mu)\Im\left(\gamma_n^\circ-\gamma_{n-1}^\circ\right)e_1(x)\,d\mu + \frac{b_n}{n}x^2\int_{-\infty}^{\infty}\phi(\mu)O(\gamma_n^\circ-\gamma_{n-1}^\circ)^2\,d\mu\right] \\ &= O(n^{-1/2}) + O(n^{-1}). \end{aligned}$$

Therefore

$$\mathbb{E}[e_1(x)] = Z_n(\phi_\eta) + O(n^{-1/2}). \tag{27}$$

Combining (18), (24), (26), and (27), we get

$$\begin{aligned} \sqrt{\frac{b_n}{n}}Y_n(z,x) &= T_1 + T_2 \\ &= \frac{2}{\mathbb{E}^2[A]}\sqrt{\frac{b_n}{n}}Y_n(z,x) + \frac{x}{2\pi}\mathbb{E}[e_1(x)]\int_{-\infty}^{\infty}[D_n(z,z_\mu)-D_n(z,\bar{z}_\mu)]\phi(\mu)\,d\mu + O(b_n^{-1/2}) \\ &= \frac{2}{\mathbb{E}^2[A]}\sqrt{\frac{b_n}{n}}Y_n(z,x) + \frac{x}{2\pi}Z_n(\phi_\eta)\int_{-\infty}^{\infty}[D_n(z,z_\mu)-D_n(z,\bar{z}_\mu)]\phi(\mu)\,d\mu + o(1) \end{aligned}$$

$$\approx 2f^2(z)\tilde{Y}_n(z,x) + \frac{x}{2\pi}Z_n(\phi_\eta)\int_{-\infty}^{\infty}[D_n(z,z_\mu)-D_n(z,\bar{z}_\mu)]\phi(\mu)\,d\mu + o(1),$$

where $\tilde{Y}_n(z,x) = \sqrt{\frac{b_n}{n}}Y_n(z,x)$. Therefore,

$$\tilde{Y}_n(z,x) = Z_n(\phi_\eta)\frac{x}{2\pi}\int_{-\infty}^{\infty}\left(C_n(z,z_\mu)-C_n(z,\bar{z}_\mu)\right)\phi(\mu)\,d\mu + o(1)$$

uniformly in z with $\Im z = \eta$, where $C_n(z,z_\mu) = \frac{D_n(z,z_\mu)}{1-2f^2(z)}$ and $f(z)$ is given in (41). Hence

$$\begin{aligned} \frac{d}{dx}Z_n(\phi_\eta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\mu)\left(\tilde{Y}_n(z_\mu,x)-\tilde{Y}_n(\bar{z}_\mu,x)\right)\,d\mu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\mu_1)\left[\frac{x}{2\pi}Z_n(\phi_\eta)\int_{-\infty}^{\infty}\phi(\mu_2)\left(C_n(z_{\mu_1},z_{\mu_2})-C_n(z_{\mu_1},\bar{z}_{\mu_2})\right)d\mu_2\right. \\ &\quad \left.-\frac{x}{2\pi}Z_n(\phi_\eta)\int_{-\infty}^{\infty}\phi(\mu_2)\left(C_n(\bar{z}_{\mu_1},z_{\mu_2})-C_n(\bar{z}_{\mu_1},\bar{z}_{\mu_2})\right)d\mu_2\right]d\mu_1 + o(1) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{x}{4\pi^2} Z_n(\phi_\eta) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\mu_1)\phi(\mu_2)\left[C_n(z_{\mu_1},\bar z_{\mu_2}) + C_n(\bar z_{\mu_1}, z_{\mu_2})\right.\\
&\quad \left. -C_n(z_{\mu_1}, z_{\mu_2}) - C_n(\bar z_{\mu_1},\bar z_{\mu_2})\right] d\mu_2 d\mu_1 + o(1)\\
&= -xZ_n(\phi_\eta)V_n(\phi,\eta) + o(1).
\end{aligned}
$$

To find the limit of $V_n(\phi,\eta)$, we shall calculate limit of $[C_n(z_{\mu_1},\bar z_{\mu_2})+C_n(\bar z_{\mu_1}, z_{\mu_2})-C_n(z_{\mu_1}, z_{\mu_2})-C_n(\bar z_{\mu_1},\bar z_{\mu_2})]$ as $n\to\infty$. Using (34) and (35),

$$
\begin{aligned}
D_n(z,z_\mu) &= \frac{(1+\mathbb{E}[B(z_\mu)])\mathbb{E}[b_n\mathbb{E}_1\{A^\circ(z)A^\circ(z_\mu)\}]}{\mathbb{E}^2[A(z)]\mathbb{E}^2[A(z_\mu)]} - \frac{\mathbb{E}[b_n\mathbb{E}_1\{A^\circ(z)B^\circ(z_\mu)\}]}{\mathbb{E}^2[A(z)]\mathbb{E}[A(z_\mu)]}\\
&= \frac{1+\mathbb{E}[B(z_\mu)]}{\mathbb{E}^2[A(z)]\mathbb{E}^2[A(z_\mu)]}\mathbb{E}\Big[\frac{2}{b_n}\sum_{i,j\in I_1} G^{(1)}_{ij}(z)G^{(1)}_{ij}(z_\mu) + \sigma^2 + \frac{\kappa_4}{b_n}\sum_{i\in I_1} G^{(1)}_{ii}(z)G^{(1)}_{ii}(z_\mu)\\
&\quad +\frac{1}{b_n}\widetilde{\gamma_{n-1}}(z)\widetilde{\gamma_{n-1}}(z_\mu)])\Big] - \frac{1}{\mathbb{E}^2[A(z)]\mathbb{E}[A(z_\mu)]}\frac{d}{dz_\mu}\mathbb{E}\Big[\frac{2}{b_n}\sum_{i,j\in I_1} G^{(1)}_{ij}(z)G^{(1)}_{ij}(z_\mu)\\
&\quad +\sigma^2 + \frac{\kappa_4}{b_n}\sum_{i\in I_1} G^{(1)}_{ii}(z)G^{(1)}_{ii}(z_\mu) + \frac{1}{b_n}\widetilde{\gamma_{n-1}}(z)\widetilde{\gamma_{n-1}}(z_\mu)])\Big].
\end{aligned}
$$

Now using (36), we get

$$
\left|\mathbb{E}\left[\frac{1}{b_n}\widetilde{\gamma_{n-1}}(z)\widetilde{\gamma_{n-1}}(z_\mu)])\right]\right| \le \frac{1}{b_n}\sqrt{\operatorname{Var}\sum_{i\in I_1} G^{(1)}_{ii}}\sqrt{\operatorname{Var}\sum_{i\in I_1} G^{(1)}_{ii}} = O\left(\frac{1}{b_n}\right).
$$

Letting $n\to\infty$, using (42) we have

$$
\begin{aligned}
\lim_{n\to\infty} D_n(z,z_\mu) &= f^2(z)f^2(z_\mu)(1+2f'(z_\mu))\left[\lim_{n\to\infty}\mathbb{E}[T_n] + \sigma^2 + \kappa_4\lim_{n\to\infty}\frac{1}{b_n}\sum_{i\in I_1}\mathbb{E}\left[G^{(1)}_{ii}(z)G^{(1)}_{ii}(z_\mu)\right]\right]\\
&\quad + f^2(z)f(z_\mu)\frac{d}{dz_\mu}\left[\lim_{n\to\infty}\mathbb{E}[T_n] + \kappa_4\lim_{n\to\infty}\frac{1}{b_n}\sum_{i\in I_1}\mathbb{E}\left[G^{(1)}_{ii}(z)G^{(1)}_{ii}(z_\mu)\right]\right],
\end{aligned}
\tag{28}
$$

where

$$
T_n = \frac{2}{b_n}\sum_{i,j\in I_1} G^{(1)}_{ij}(z)G^{(1)}_{ij}(z_\mu).
$$

Since $\operatorname{Var}(G_{ii}) = O(1/b_n)$ (see (36)), we have

$$
\lim_{n\to\infty}\frac{1}{b_n}\sum_{i\in I_1}\mathbb{E}\left[G^{(1)}_{ii}(z)G^{(1)}_{ii}(z_\mu)\right] = \lim_{n\to\infty}\frac{1}{b_n}\sum_{i\in I_1}\mathbb{E}\left[G^{(1)}_{ii}(z)\right]\mathbb{E}\left[G^{(1)}_{ii}(z_\mu)\right] = 2f(z)f(z_\mu).
$$

We shall show in the appendix (4.1) that

$$
\lim_{n\to\infty}\mathbb{E}[T_n] = \frac{1}{4\pi^3}\int_{-2\sqrt2}^{2\sqrt2}\int_{-2\sqrt2}^{2\sqrt2}\frac{\sqrt{8-x^2}\sqrt{8-y^2}}{(x-z)(y-z_\mu)}F(x,y)\mathbf{1}_{\{x\neq y\}}\,dxdy,
\tag{29}
$$

where

$$
F(x,y) = 2\int_{-\infty}^{\infty}\frac{u-u^3}{2(1-u^2)^2 + u^2(x^2+y^2) - u(1+u^2)xy}\,ds,
$$

where $u = \frac{\sin s}{s}$. Therefore

$$
\lim_{n\to\infty} C_n(z_{\mu_1}, z_{\mu_2}) = \frac{1}{1-2f^2(z_{\mu_1})}\left[f^2(z_{\mu_1})f^2(z_{\mu_2})(1+2f'(z_{\mu_2}))\lim_{n\to\infty}\mathbb{E}[T_n] + f^2(z_{\mu_1})f(z_{\mu_2})\lim_{n\to\infty}\frac{d}{dz_{\mu_2}}\mathbb{E}[T_n]\right.
$$

$$
+\sigma^2 f^2(z_{\mu_1}) f^2(z_{\mu_2})(1+2f'(z_{\mu_2})) + 2\kappa_4 \Big\{ f^3(z_{\mu_1}) f^3(z_{\mu_2})(1+2f'(z_{\mu_2}))
+ f^3(z_{\mu_1}) f(z_{\mu_2}) f'(z_{\mu_2}) \Big\} \Big].
$$

Hence

$$
\begin{aligned}
V(\phi) &= \lim_{\eta \downarrow 0} \lim_{n\to\infty} V_n(\phi,\eta) \\
&= \frac{\kappa_4}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{4-\mu^2}{\sqrt{8-\mu^2}} \phi(\mu)\, d\mu \right)^2 + \frac{\sigma^2}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu \phi(\mu)}{\sqrt{8-\mu^2}}\, d\mu \right)^2 \\
&\quad + \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{(8-x^2)(8-y^2)} F(x,y) \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu_1 \phi(\mu_1)}{(x-\mu_1)\sqrt{8-\mu_1^2}} \frac{\mu_2\phi(\mu_2)}{(x-\mu_2)^2\sqrt{8-\mu_2^2}} d\mu_1 d\mu_2\; dxdy.
\end{aligned}
$$

This completes the proof of (10) and the proof of Theorem 1. □

4 Appendix

Proof of Proposition 2: Let us denote the averaging with respect to $\{w_{ij}; 1 \le i \le k \text{ or } 1 \le j \le n\}$ by $\mathbb{E}_{\le k}$ and the averaging with respect to $\{w_{kj}; 1 \le j \le n\}$ by $\mathbb{E}_k$. Using the martingale difference technique (see [**?**]), we have

$$
\begin{aligned}
\mathrm{Var}\{\gamma_n\} &\le \sum_{k=1}^n \mathbb{E}\left[\left| \mathbb{E}_{\le k-1}[\gamma_n] - \mathbb{E}_{\le k}[\gamma_n] \right|^2 \right] \\
&= \sum_{k=1}^n \mathbb{E}\left[\left| \mathbb{E}_{\le k-1}\left[\gamma_n - \mathbb{E}_{\le k}[\gamma_n]\right] \right|^2 \right] \\
&\le \sum_{k=1}^n \mathbb{E}\left[\mathbb{E}_{\le k-1} \left| \gamma_n - \mathbb{E}_k[\gamma_n] \right|^2 \right] \\
&= \sum_{k=1}^n \mathbb{E}\left[\left| \gamma_n - \mathbb{E}_k[\gamma_n] \right|^2 \right].
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathbb{E}\left[|\gamma_n - \mathbb{E}_1[\gamma_n]|^2 \right] &= \mathbb{E}\left[|\mathrm{Tr}(G) - \mathbb{E}_1[\mathrm{Tr}(G)]|^2 \right] \\
&= \mathbb{E}\left[\left| \mathrm{Tr}(G) - \mathbb{E}_1[\mathrm{Tr}(G)] + \mathrm{Tr}(G^{(1)}) - \mathrm{Tr}(G^{(1)}) \right|^2 \right] \\
&= \mathbb{E}\left[\left| \mathrm{Tr}(G - G^{(1)}) - \mathbb{E}_1\left[\mathrm{Tr}(G - G^{(1)}) \right] \right|^2 \right].
\end{aligned}
$$

From (32) we have

$$
\mathrm{Tr}(G - G^{(1)}) = -\frac{1+B(z)}{A(z)} \tag{30}
$$

where $A(z) = -G_{11}^{-1}$, $B(z) = \langle G^{(1)} G^{(1)} m^{(1)}, m^{(1)} \rangle$, and $G^{(1)}$ is defined in (16), and $m^{(1)} = \frac{1}{\sqrt{b_n}}(w_{12}, w_{13}, \ldots, w_{1n})^T$. Indeed,

$$
\mathbb{E}\left[|\gamma_n - \mathbb{E}_1[\gamma_n]|^2 \right] \le \mathbb{E}\left[\left| \frac{1+B(z)}{A(z)} - \mathbb{E}_1\left[\frac{1+B(z)}{A(z)} \right] \right|^2 \right]
$$

$$\leq \quad 2\mathbb{E}\left[\left|\frac{1}{A(z)} - \mathbb{E}_1\left[\frac{1}{A(z)}\right]\right|^2\right] + 2\mathbb{E}\left[\left|\frac{B(z)}{A(z)} - \mathbb{E}_1\left[\frac{B(z)}{A(z)}\right]\right|^2\right].$$

Now, by (21) and (25),

$$\begin{aligned}
\mathbb{E}_1\left[\left|\frac{B(z)}{A(z)} - \mathbb{E}_1\left[\frac{B(z)}{A(z)}\right]\right|^2\right] &\leq \mathbb{E}_1\left[\left|\frac{B(z)}{A(z)} - \frac{\mathbb{E}_1[B(z)]}{E_1[A(z)]}\right|^2\right] \\
&\leq \mathbb{E}_1\left[\left|\frac{B_1^\circ}{\mathbb{E}_1[A]} - \frac{A_1^\circ}{\mathbb{E}_1[A]}\frac{B}{A}\right|^2\right] \\
&\leq 2\mathbb{E}_1\left[\left|\frac{B_1^\circ}{\mathbb{E}_1[A]}\right|^2\right] + \frac{2}{|\Im z|^2}\mathbb{E}_1\left[\left|\frac{A_1^\circ}{\mathbb{E}_1[A]}\right|^2\right],
\end{aligned}$$

where $A_1^\circ = A - \mathbb{E}_1[A]$. So it is enough to estimate $\mathbb{E}_1\left[\left|\frac{A_1^\circ}{\mathbb{E}_1[A]}\right|^2\right]$ and $\mathbb{E}_1\left[\left|\frac{B_1^\circ}{\mathbb{E}_1[A]}\right|^2\right]$. Note that

$$\begin{aligned}
A &= z - \frac{1}{\sqrt{b_n}}w_{11} + \left\langle G^{(1)}m^{(1)}, m^{(1)}\right\rangle \\
A_1^\circ &= -\frac{1}{\sqrt{b_n}}w_{11} + \frac{1}{b_n}\sum_{\substack{i\neq j\\ i,j\in I_1}} G_{ij}^{(1)}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1} G_{ii}^{(1)}(w_{1i}^2)^\circ.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_1\left[|A_1^\circ|^2\right] &= \mathbb{E}_1\left[\frac{1}{b_n}w_{11}^2 + \frac{1}{b_n^2}\sum_{\substack{i\neq j\\ i,j\in I_1}} G_{ij}^{(1)}w_{1i}w_{1j}\sum_{\substack{k\neq l\\ k,l\in I_1}}\overline{G_{kl}^{(1)}}w_{1k}w_{1l} + \frac{1}{b_n^2}\sum_{i\in I_1}G_{ii}^{(1)}(w_{1i}^2)^\circ\sum_{l\in I_1}\overline{G_{ll}^{(1)}}(w_{1l}^2)^\circ\right] \\
&= \frac{\sigma^2}{b_n} + \frac{2}{b_n^2}\sum_{\substack{i\neq j\\ i,j\in I_1}}|G_{ij}^{(1)}|^2 + \frac{\mu_4 - 1}{b_n^2}\sum_{i\in I_1}|G_{ii}^{(1)}|^2 \\
&\leq \frac{\sigma^2}{b_n} + \frac{2}{b_n^2}\frac{2b_n}{|\Im z|^2} + \frac{\mu_4-1}{b_n^2}\frac{2b_n}{|\Im z|^2} \\
&\leq \frac{1}{b_n}\left(\sigma^2 + \frac{2+2\mu_4}{|\Im z|^2}\right). \qquad (31)
\end{aligned}$$

Now we want to estimate $\mathbb{E}_1\left[|B_1^\circ|^2\right]$, where $B = \left\langle G^{(1)}G^{(1)}m^{(1)}, m^{(1)}\right\rangle = \left\langle H^{(1)}m^{(1)}, m^{(1)}\right\rangle$, and $B_1^\circ = B - \mathbb{E}_1[B]$. Therefore,

$$\mathbb{E}_1[B] = \frac{1}{b_n}\sum_{i\in I_1}H_{ii}^{(1)} = \frac{1}{b_n}\sum_{i\in I_1}\sum_{j=2}^{n}\left(G_{ij}^{(1)}\right)^2,$$

and

$$B_1^\circ = \frac{1}{b_n}\sum_{\substack{i\neq j\\ i,j\in I_1}}H_{ij}^{(1)}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1}H_{ii}^{(1)}\left(w_{1i}^2\right)^\circ.$$

Let us call $C_0 = \mathbb{E}\left[(w_{1i}^2)^\circ\right]^2$. Then

$$\mathbb{E}_1[|B_1^\circ|^2] = \frac{1}{b_n^2}\sum_{\substack{i\neq j\\ i,j\in I_1}}|H_{ij}^{(1)}|^2 + \frac{C_0}{b_n^2}\sum_{i\in I_1}|H_{ii}^{(1)}|^2$$

$$
\begin{aligned}
&= \frac{1}{b_n^2} \sum_{\substack{i\neq j\\ i,j\in I_1}} \left| \sum_{k=2}^n G^{(1)}_{ik} G^{(1)}_{kj} \right|^2 + \frac{C_0}{b_n^2} \sum_{i\in I_1} \left| \sum_{k=2}^n G^{(1)}_{ik} G^{(1)}_{ki} \right|^2 \\
&\leq \frac{1}{b_n^2} \sum_{i\in I_1} \|(G^{(1)})^2\|^2 + \frac{C_0}{b_n^2} 2b_n \|(G^{(1)})^2\|^2 \\
&= \frac{2}{b_n}\frac{1}{|\Im z|^4} + \frac{2C_0}{b_n}\frac{1}{|\Im z|^4} \\
&= \frac{2(1+C_0)}{b_n|\Im z|^4}.
\end{aligned}
$$

We also have

$$
\mathbb{E}\left[\left|\frac{1}{A(z)} - \mathbb{E}_1\left[\frac{1}{A(z)}\right]\right|^2\right] \leq \frac{1}{|\Im z|^2}\mathbb{E}\left[\left|\frac{A_1^\circ}{\mathbb{E}_1[A]}\right|^2\right]
$$

Note that $\mathbb{E}_1[A] = z + \frac{1}{b_n}\sum_{i\in I_1} G^{(1)}_{ii}$. Since $\Im G^{(1)}_{ii} > 0$, we have $|\mathbb{E}_1[A]| \geq |\Im A| \geq y$. Also we know that $|G^{(1)}_{ii}| \leq 1/|\Im z| = 1/y$. Therefore $|\mathbb{E}_1[A]| \geq |x| - \frac{2}{y}$. Combining these we have

$$
|\mathbb{E}_1[A]| > \max\left\{y, |x| - \frac{2}{y}\right\}.
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[|\gamma_n - \mathbb{E}_1[\gamma_n]|^2\right] &\leq C_1\frac{2(1+C_0)}{b_n|\Im z|^4}|\mathbb{E}_1[A]|^{-2} + \frac{C_2}{b_n}\left(\sigma^2 + \frac{2+2\mu_4}{|\Im z|^2}\right)\frac{|\mathbb{E}_1[A]|^{-2}}{|\Im z|^2} \\
&\leq \frac{C}{b_n}\left(\frac{1}{|\Im z|^2} + \frac{1}{|\Im z|^4}\right)|\mathbb{E}_1[A]|^{-2},
\end{aligned}
$$

for some $C_1, C_2, C > 0$ not depending on z, n. This implies

$$
\mathrm{Var}(\gamma_n) \leq \frac{Cn}{b_n}\left(\frac{1}{|\Im z|^2} + \frac{1}{|\Im z|^4}\right)\left(\max\left\{y, |x| - \frac{2}{y}\right\}\right)^{-2}.
$$

This completes the proof of proposition 2. $\square$

Now we proceed to the proofs of the asymptotic estimates. All the asymptotic estimates listed in Lemma 1 and Lemma 2 hold uniformly in the set $\{z \in \mathbb{C} : |\Im z| \geq \eta\}$ for any given $\eta > 0$.

Lemma 1. *Let M be an $n \times n$ symmetric band matrix as defined in* (2) *which satisfies* (3) *and $\mathbb{E}[|w_{ij}|^8]$ is uniformly bounded. Then*

(i)

$$
G^{(1)}_{ii} - G_{ii} = \frac{1}{A(z)}\left(G^{(1)}m^{(1)}\right)_i^2 = \frac{1}{A(z)}\left(\frac{1}{\sqrt{b_n}}\sum_{j\in I_1} G^{(1)}_{ij} w_{1j}\right)^2 \tag{32}
$$

where $2 \leq i \leq n$, $A(z)$, $m^{(1)}$ and $G^{(1)}$ are as defined in (14), (15) *and* (16).

(ii) $\left|\mathbb{E}\left[G^{(1)}_{ii}(z)\right] - \mathbb{E}[G_{ii}(z)]\right| = O\left(\frac{1}{b_n}\right)$.

(iii)

$$
\mathbb{E}[|G_{12}|^2] = O\left(\frac{1}{b_n}\right)\frac{1}{|\Im z|^6},\ \mathbb{E}[|G_{12}|^4] = O\left(\frac{1}{b_n^2}\right)\frac{1}{|\Im z|^{12}} \text{ and } \mathbb{E}[|G_{12}|^8] = O\left(\frac{1}{b_n^4}\right)\frac{1}{|\Im z|^{24}}. \tag{33}
$$

(iv) Let us denote the averaging with respect to $\{w_{1i}\}_{1\le i\le n}$ by $\mathbb{E}_1$. Then

$$b_n\mathbb{E}_1\left[A^\circ(z_1)A^\circ(z_2)\right] = \sigma^2 + \frac{2}{b_n}\sum_{i,j\in I_1} G_{ij}^{(1)}(z_1)G_{ij}^{(1)}(z_2) + \frac{\kappa_4}{b_n}\sum_{i\in I_1} G_{ii}^{(1)}(z_1)G_{ii}^{(1)}(z_2) + \frac{1}{b_n}\widetilde{\gamma_{n-1}}(z_1)\widetilde{\gamma_{n-1}}(z_2) \tag{34}$$

where $\widetilde{\gamma_{n-1}}(z) = \sum_{i\in I_1}\left(G_{ii}^{(1)} - \mathbb{E}[G_{ii}^{(1)}(z)]\right)$ *and* $I_1 = \{1 < i \le n : (1,i) \in I_n\}$.

(v)

$$\mathbb{E}_1\left[A^\circ(z_1)B^\circ(z_2)\right] = \frac{d}{dz_2}\mathbb{E}_1\left[A^\circ(z_1)A^\circ(z_2)\right] \;\; \textit{where } B(z_2) = \left\langle G^{(1)}(z_2)G^{(1)}(z_2)m^{(1)}, m^{(1)}\right\rangle. \tag{35}$$

Lemma 2. *Let M be an $n\times n$ symmetric band matrix as defined in (2) which satisfies (3). Also assume that the probability distribution of w_{jk} satisfies the Poincaré inequality with some uniform constant m which does not depend on n, j, k. Then*

(i)

$$Var\Big(\sum_{(1,i)\in I_n} G_{ii}\Big) = O(1) \quad \textit{and} \quad Var(G_{11}(z)) = O\left(\frac{1}{b_n}\right). \tag{36}$$

(ii)

$$\mathbb{E}\left[|A^\circ|^4\right] = O\left(\frac{1}{b_n^2}\right), \qquad \mathbb{E}\left[|A^\circ|^3\right] = O\left(\frac{1}{b_n^{3/2}}\right) \tag{37}$$

$$\mathbb{E}\left[|B^\circ|^4\right] \;=\; O\left(\frac{1}{b_n^2}\right) \tag{38}$$

(iii)

$$Var\{b_n\mathbb{E}_1\left[A^\circ(z_1)A^\circ(z_2)\right]\} = O\left(\frac{1}{b_n}\right) \;\textit{and}\; Var\{b_n\mathbb{E}_1\left[A^\circ(z_1)B^\circ(z_2)\right]\} = O\left(\frac{1}{b_n}\right) \tag{39}$$

(iv)

$$\mathbb{E}\left[\left|\gamma_{n-1}^\circ(z) - \gamma_n^\circ(z)\right|^4\right] = O\left(\frac{1}{b_n^2}\right) \quad \textit{and} \;\; \mathbb{E}\left[|\gamma_n^\circ|^4\right] = O\left(\frac{n^2}{b_n^2}\right). \tag{40}$$

(v)

$$\frac{1}{n}\mathbb{E}\left[TrG(z)\right] = f(z) + O\left(\frac{1}{|\Im z|^6 b_n}\right) \;\textit{where } f(z) = \frac{1}{4}\left(-z + \sqrt{z^2-8}\right). \tag{41}$$

(vi)

$$(\mathbb{E}[A(z)])^{-1} = -f(z) + O(b_n^{-1}) \quad \textit{and} \;\; \mathbb{E}[B(z)] = 2f'(z) + O(b_n^{-1}). \tag{42}$$

Proof of Lemma 1: **Proof of (i):** Suppose $(X_1, X_2, \ldots, X_n)$ is a n dimensional normal random vector with a positive definite covariance matrix A^{-1} and a mean $A^{-1}\underline{h}$, where $\underline{h}\in\mathbb{R}^n$. Then we have

$$\int \exp\left[-\frac{1}{2}\langle A\underline{x},\underline{x}\rangle + \langle \underline{h},\underline{x}\rangle\right]\, d\underline{x} \;=\; (2\pi)^{n/2}|\det A|^{-1/2}\exp\left[\frac{1}{2}\langle A^{-1}\underline{h},\underline{h}\rangle\right], \tag{43}$$

$$\frac{\int x_ix_j\exp\left[-\frac{1}{2}\langle A\underline{x},\underline{x}\rangle + \langle \underline{h},\underline{x}\rangle\right]\, d\underline{x}}{\int \exp\left[-\frac{1}{2}\langle A\underline{x},\underline{x}\rangle + \langle \underline{h},\underline{x}\rangle\right]\, d\underline{x}} \;=\; (A^{-1})_{ij} + (A^{-1}\underline{h})_i(A^{-1}\underline{h})_j. \tag{44}$$

where $\underline{x} = (x_1, x_2, \ldots, x_n)^T$. In particular, for $\underline{h} = 0$,

$$(A^{-1})_{ij} = \frac{\int x_ix_j\exp[-\frac{1}{2}\langle A\underline{x},\underline{x}\rangle]\, d\underline{x}}{\int \exp[-\frac{1}{2}\langle A\underline{x},\underline{x}\rangle]\, d\underline{x}}. \tag{45}$$

Now doing the integrations in (45) with respect to all variables except x_1, and using (43) we get

$$\begin{aligned}\int \exp[-\frac{1}{2}\langle A\underline{x},\underline{x}\rangle]\; d\underline{x} &= \int \exp[-\frac{a_{11}x_1^2}{2}] \int \exp[-\frac{1}{2}\langle A_1\underline{x}^{(1)},\underline{x}^{(1)}\rangle - \langle x_1\underline{a_1},\underline{x}^{(1)}\rangle]\; d\underline{x} \\ &= \frac{(2\pi)^{\frac{n-1}{2}}}{|\det A_1|^{1/2}} \int \exp[-\frac{x_1^2}{2}(a_{11} - \langle A_1^{-1}\underline{a_1},\underline{a_1}\rangle)]\; dx_1\end{aligned}$$

where $\underline{x}^{(1)} = (x_2, x_3, \ldots, x_n)^T$, $\underline{a_1} = (a_{12}, a_{13}, \ldots, a_{1n})^T$ and $A_1 = ((A_1)_{ij})_{i,j=2}^n$ is the $(n-1)\times(n-1)$ matrix obtained from A after removing first row and first column, and for $i, j \neq 1$, using (44) and (43) we get

$$\begin{aligned}&\int x_i x_j \exp[-\frac{1}{2}\langle A\underline{x},\underline{x}\rangle]\; d\underline{x} \\ = &\int \exp[-\frac{a_{11}x_1^2}{2}] \int x_i x_j \exp[-\frac{1}{2}\langle A_1\underline{x}^{(1)},\underline{x}^{(1)}\rangle - \langle x_1\underline{a_1},\underline{x}^{(1)}\rangle]\; d\underline{x}^{(1)}dx_1 \\ = &\int \exp[-\frac{a_{11}x_1^2}{2}]\; [(A_1^{-1})_{ij} + x_1^2(A_1^{-1}\underline{a_1})_i(A_1^{-1}\underline{a_1})_j] \int \exp[-\frac{1}{2}\langle A_1\underline{x}^{(1)},\underline{x}^{(1)}\rangle - \langle x_1\underline{a_1},\underline{x}^{(1)}\rangle]\; d\underline{x}^{(1)}dx_1 \\ = &\frac{(2\pi)^{\frac{n-1}{2}}}{|\det A_1|^{1/2}} \int [(A_1^{-1})_{ij} + x_1^2(A_1^{-1}\underline{a_1})_i(A_1^{-1}\underline{a_1})_j]\; \exp[-\frac{x_1^2}{2}(a_{11} - \langle A_1^{-1}\underline{a_1},\underline{a_1}\rangle)]\; dx_1.\end{aligned}$$

Therefore, from (45) we get

$$\begin{aligned}(A^{-1})_{ij} &= (A_1^{-1})_{ij} + (A_1^{-1}\underline{a_1})_i(A_1^{-1}\underline{a_1})_j \;\frac{\int x_1^2 \exp[-\frac{x_1^2}{2}(a_{11} - \langle A_1^{-1}\underline{a_1},\underline{a_1}\rangle)]\; dx_1}{\int \exp[-\frac{x_1^2}{2}(a_{11} - \langle A_1^{-1}\underline{a_1},\underline{a_1}\rangle)]\; dx_1} \\ &= (A_1^{-1})_{ij} + \frac{(A_1^{-1}\underline{a_1})_i(A_1^{-1}\underline{a_1})_j}{a_{11} - \langle A_1^{-1}\underline{a_1},\underline{a_1}\rangle}.\end{aligned}$$

Applying the above formula for $A = (M - zI)$, where $z \in \mathbb{R}$, $|z| > \|M\|$, we obtain

$$G_{ij} = G_{ij}^{(1)} + \frac{(G^{(1)}m^{(1)})_i(G^{(1)}m^{(1)})_j}{\frac{w_{11}}{\sqrt{b_n}} - z - \langle G^{(1)}m^{(1)}, m^{(1)}\rangle}, \quad i, j \geq 2,$$

where $m^{(1)}$, $G^{(1)}$ are as defined in (15), (16) respectively. From the above formula we obtain

$$G_{ii} - G_{ii}^{(1)} = -\frac{(G^{(1)}m^{(1)})_i^2}{A(z)}, \quad \text{for all } \; 2 \leq i \leq n,$$

where $A(z)$ is as defined in (14). The above is true for all $z \in \mathbb{R}$ such that $|z| > \|M\|$. By analytic continuity one can extend it to the whole complex plane. This completes the proof.

Proof of (ii): Recall $I_1 = \{1 < i \leq n : (1, i) \in I_n\}$. Now using (32) and (21) we have

$$\begin{aligned}\left|\mathbb{E}\left[G_{ii}^{(1)}(z)\right] - \mathbb{E}[G_{ii}(z)]\right| &= \left|\mathbb{E}\left[\frac{1}{A}\left(G^{(1)}m^{(1)}\right)_i^2\right]\right| \\ &= \left|\mathbb{E}\left[\frac{1}{A}\left(\frac{1}{\sqrt{b_n}}\sum_{j\in I_1} G_{ij}^{(1)}w_{1j}\right)^2\right]\right| \\ &\leq \frac{1}{b_n}\frac{1}{|\Im z|}\mathbb{E}\left[\left|\sum_{j\in I_1} G_{ij}^{(1)}w_{1j}\right|^2\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{b_n|\Im z|}\mathbb{E}\left[\sum_{j\in I_1}|G_{ij}^{(1)}|^2 w_{1j}^2 + \sum_{j_1\neq j_2\in I_1} G_{ij_1}^{(1)}\overline{G_{ij_2}^{(1)}} w_{1j_1}w_{1j_2}\right]\\
&= \frac{1}{b_n|\Im z|}\mathbb{E}\mathbb{E}_1\left[\sum_{j\in I_1}|G_{ij}^{(1)}|^2 w_{1j}^2 + \sum_{j_1\neq j_2\in I_1} G_{ij_1}^{(1)}\overline{G_{ij_2}^{(1)}} w_{1j_1}w_{1j_2}\right]\\
&= \frac{1}{b_n|\Im z|}\mathbb{E}\left[\sum_{j\in I_1}|G_{1j}^{(1)}|^2\right]\\
&\leq \frac{1}{b_n|\Im z|}\mathbb{E}\|G^{(1)}\|^2 \leq \frac{1}{b_n|\Im z|^3}.
\end{aligned}
$$

Proof of (iii): Using the resolvent formula given in [?], we have

$$G_{12} = -G_{22}G_{11}^{(2)}K_{12}^{(12)},$$

where $G^{(2)}$ is the resolvent of the $(n-1)\times(n-1)$ minor obtained by removing the kth row and kth column from the matrix M, $K_{12}^{(12)} = m_{12} - m_{(1)}G^{(12)}m_{(2)}$, $m_{(1)} = \frac{1}{\sqrt{b_n}}(w_{13}, w_{14}, \ldots, w_{1n})$, $m_{(2)} = \frac{1}{\sqrt{b_n}}(w_{23}, w_{24}, \ldots, w_{2n})^T$, $G^{(ij)} = \left(M^{(ij)} - zI\right)^{-1}$, and $M^{(ij)}$ is $(n-2)\times(n-2)$ matrix obtained from M after removing ith and jth rows and columns. Therefore,

$$
\begin{aligned}
\mathbb{E}[|G_{12}|^2] &= \mathbb{E}\left[\left|G_{22}G_{11}^{(2)}K_{12}^{(12)}\right|^2\right]\\
&\leq \frac{1}{|\Im z|^2}\frac{1}{|\Im z|^2}\mathbb{E}\left[\left|m_{12} - m_{(1)}G^{(12)}m_{(2)}\right|^2\right]\\
&= \frac{1}{|\Im z|^4}\mathbb{E}\left[\left|\frac{w_{12}}{\sqrt{b_n}} - \frac{1}{b_n}\sum_{\substack{(1,i),(2,j)\in I_n\\ i,j\neq 1,2}} G_{ij}^{(12)} w_{1i}w_{2j}\right|^2\right]\\
&\leq \frac{1}{|\Im z|^4}\mathbb{E}\mathbb{E}_{\leq 2}\left[\frac{w_{12}^2}{b_n} + \frac{1}{b_n^2}\sum_{\substack{(1,i),(2,j)\in I_n\\ i,j\neq 1,2}} |G_{ij}^{(12)}|^2 w_{1i}^2 w_{2j}^2\right]\\
&\leq \frac{1}{|\Im z|^4}\mathbb{E}\left[\frac{1}{b_n} + \frac{1}{b_n^2}\sum_{i,j}|G_{ij}^{(12)}|^2\mathbb{E}_{\leq 2}[w_{1i}^2]\mathbb{E}_{\leq 2}[w_{2j}^2]\right]\\
&\leq \frac{1}{|\Im z|^4}\mathbb{E}\left[\frac{1}{b_n} + \frac{1}{b_n^2}\frac{b_n}{|\Im z|^2}\right] = O\left(\frac{1}{b_n}\right)\frac{1}{|\Im z|^6},
\end{aligned}
$$

where $E_{\leq 2}$ is the averaging with respect to the first two rows and columns. Similarly, we can prove that $\mathbb{E}[|G_{12}|^4] = O\left(\frac{1}{b_n^2}\right)\frac{1}{|\Im z|^{12}}$, and $\mathbb{E}[|G_{12}|^8] = O\left(\frac{1}{b_n^4}\right)\frac{1}{|\Im z|^{24}}$.

Proof of (iv): We know that

$$
\begin{aligned}
A(z_1) &= z_1 - \frac{w_{11}}{\sqrt{b_n}} + \left\langle G^{(1)}m^{(1)}, m^{(1)}\right\rangle,\\
\text{and}\quad A^\circ(z_1) &= -\frac{w_{11}}{\sqrt{b_n}} + \frac{1}{b_n}\sum_{i\neq j\in I_1} G_{ij}^{(1)}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1} G_{ii}^{(1)}w_{1i}^2 - \frac{1}{b_n}\sum_{i\in I_1}\mathbb{E}[G_{ii}^{(1)}].
\end{aligned}
$$

Now we can estimate

$$b_n\mathbb{E}_1\left[A^\circ(z_1)A^\circ(z_2)\right]$$

$$
\begin{aligned}
&= \sigma^2 + \frac{1}{b_n}\mathbb{E}_1\left[\sum_{\substack{i_1\neq j_1\in I_1\\ i_2\neq j_2\in I_1}} G^{(1)}_{i_1j_1}(z_1)w_{1i_1}w_{1j_1}G^{(1)}_{i_2j_2}(z_2)w_{1i_2}w_{1j_2}\right] + \frac{1}{b_n}\mathbb{E}_1\left[\sum_{i,j\in I_1} G^{(1)}_{ii}(z_1)G^{(1)}_{jj}(z_2)w_{1i}^2w_{1j}^2\right]\\
&\quad -\frac{1}{b_n}\mathbb{E}\left[\sum_{i\in I_1} G^{(1)}_{ii}(z_2)\right]\mathbb{E}_1\left[\sum_{i\in I_1} G^{(1)}_{ii}(z_1)w_{1i}^2\right] - \frac{1}{b_n}\mathbb{E}\left[\sum_{i\in I_1} G^{(1)}_{ii}(z_1)\right]\mathbb{E}_1\left[\sum_{i\in I_1} G^{(1)}_{ii}(z_2)w_{1i}^2\right]\\
&\quad +\frac{1}{b_n}\mathbb{E}\left[\sum_{i\in I_1} G^{(1)}_{ii}(z_1)\right]\mathbb{E}\left[\sum_{i\in I_1} G^{(1)}_{ii}(z_2)\right]\\
&= \sigma^2 + \frac{2}{b_n}\sum_{i\neq j\in I_1} G^{(1)}_{ij}(z_1)G^{(1)}_{ij}(z_2) + \frac{1}{b_n}\sum_{i\neq j\in I_1} G^{(1)}_{ii}(z_1)G^{(1)}_{jj}(z_2) + \frac{\mu_4}{b_n}\sum_{i\in I_1} G^{(1)}_{ii}(z_1)G^{(1)}_{ii}(z_2)\\
&\quad +\frac{1}{b_n}\widetilde{\gamma_{n-1}}(z_1)\widetilde{\gamma_{n-1}}(z_2) - \frac{1}{b_n}\left(\sum_{i\in I_1} G^{(1)}_{ii}(z_1)\right)\left(\sum_{i\in I_1} G^{(1)}_{ii}(z_2)\right)\\
&= \sigma^2 + \frac{2}{b_n}\sum_{i,j\in I_1} G^{(1)}_{ij}(z_1)G^{(1)}_{ij}(z_2) + \frac{\mu_4}{b_n}\sum_{i\in I_1} G^{(1)}_{ii}(z_1)G^{(1)}_{ii}(z_2) - \frac{3}{b_n}\sum_{i\in I_1} G^{(1)}_{ii}(z_1)G^{(1)}_{ii}(z_2)\\
&\quad +\frac{1}{b_n}\widetilde{\gamma_{n-1}}(z_1)\widetilde{\gamma_{n-1}}(z_2)\\
&= \sigma^2 + \frac{2}{b_n}\sum_{i,j\in I_1} G^{(1)}_{ij}(z_1)G^{(1)}_{ij}(z_2) + \frac{\kappa_4}{b_n}\sum_{i\in I_1} G^{(1)}_{ii}(z_1)G^{(1)}_{ii}(z_2) + \frac{1}{b_n}\widetilde{\gamma_{n-1}}(z_1)\widetilde{\gamma_{n-1}}(z_2),
\end{aligned}
$$

where $\kappa_4 = \mu_4 - 3$.

Proof of (v): Observe that

$$
B(z_2) = \left\langle G^{(1)}G^{(1)}m^{(1)}, m^{(1)}\right\rangle = \frac{1}{b_n}\sum_{i,j\in I_1}\left(G^{(1)}G^{(1)}\right)_{ij} w_{1i}w_{1j} = \frac{1}{b_n}\sum_{i,j\in I_1}\sum_{k=2}^{n} G^{(1)}_{ik}G^{(1)}_{kj}w_{1i}w_{1j},
$$

and

$$
\frac{d}{dz_2}G^{(1)}_{ij}(z_2) = \left(G^{(1)}(z_2)G^{(1)}(z_2)\right)_{ij} = \sum_{k=2}^{n} G^{(1)}_{ik}(z_2)G^{(1)}_{kj}(z_2).
$$

Now, proceed as in **(iv)** and use the above facts to prove the result. Here we skip the details.

□

Proof of Lemma 2: **Proof of (i):** Since w_{jk} satisfies the Poincaré inequality with constant m and the Poincaré inequality tensorises, the joint distribution of $\{w_{jk}\}_{(j,k)\in I_n^+}$ on $\mathbb{R}^{n(b_n+1)}$ satisfies the Poincaré inequality with same constant m. Therefore we have

$$
\mathrm{Var}\left(\Phi\left(\{w_{jk}\}_{(j,k)\in I_n^+}\right)\right) \le \frac{1}{m}\sum_{(j,k)\in I_n^+}\mathbb{E}\left[\left|\frac{\partial\Phi}{\partial w_{jk}}\right|^2\right],
$$

for any continuously differentiable function Φ. Therefore,

$$
\mathrm{Var}\left(\sum_{(1,i)\in I_n} G_{ii}\right) \le \frac{1}{m}\sum_{(j,k)\in I_n^+}\mathbb{E}\left[\left|\frac{\partial}{\partial w_{jk}}\sum_{(1,i)\in I_n} G_{ii}\right|^2\right] \tag{46}
$$

$$
\begin{aligned}
&\leq \frac{4}{mb_n}\sum_{(j,k)\in I_n^+}\mathbb{E}\left[\left|\sum_{(1,i)\in I_n} G_{ij}G_{ki}\right|^2\right] \\
&= \frac{4}{mb_n}\sum_{(j,k)\in I_n^+}\mathbb{E}\left[|\alpha_{kj}|^2\right] \quad \text{where } \alpha_{kj} = \sum_{(1,i)\in I_n} G_{ki}G_{ij} \\
&\leq \frac{4}{mb_n}\sum_{j,k=1}^{n}\mathbb{E}\left[|\alpha_{kj}|^2\right] \\
&= \frac{4}{mb_n}\mathbb{E}\left[\|VV^T\|_{Fb}^2\right] \\
&= \frac{4}{mb_n}\mathbb{E}\left[\sum_{i=1}^{n}|\beta_i|^2\right],
\end{aligned}
$$

where

$$
V = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1k_n} & 0 & \cdots & 0 \\ G_{21} & G_{22} & \cdots & G_{2k_n} & 0 & \cdots & 0 \\ & & & \vdots & & & \\ G_{n1} & G_{n2} & \cdots & G_{nk_n} & 0 & \cdots & 0 \end{bmatrix}_{n\times n}
$$

and $\|\cdot\|_{Fb}$ stands for the Frobenius norm, and β_is are the eigenvalues of VV^T. Here, we denote the set $\{i : (1,i) \in I_n\}$ by $\{1,2,\ldots,k_n\}$. Observe that $k_n = 2b_n + 1$. Since $rank(VV^T) \leq k_n = O(b_n)$, we have $\#\{i : \beta_i \neq 0\} \leq k_n = O(b_n)$. Also we know that $\|V\| \leq \|G\|$. Therefore,

$$
|\beta_i|^2 \leq \|VV^T\|^2 \leq \|G\|^4 \leq \frac{1}{|\Im z|^4}.
$$

Consequently, we have

$$
\mathrm{Var}\left(\sum_{(1,i)\in I_n} G_{ii}\right) \leq \frac{4}{mb_n}\mathbb{E}\left[\sum_{i=1}^{n}|\beta_i|^2\right] \leq \frac{4}{mb_n}\frac{O(b_n)}{|\Im z|^4} = O(1). \tag{47}
$$

This completes proof of first part of (36).

Recall the definition of A from (14), $A = z - \frac{1}{\sqrt{b_n}}w_{11} + \left(G^{(1)}m^{(1)}, m^{(1)}\right)$. Then

$$
\begin{aligned}
A^\circ &= A - \mathbb{E}[A] \\
&= -\frac{1}{\sqrt{b_n}}w_{11} + \frac{1}{b_n}\sum_{\substack{i\neq j\\ i,j\in I_1}} G_{ij}^{(1)}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1}\left(G_{ii}^{(1)}w_{1i}^2 - \mathbb{E}[G_{ii}^{(1)}]\right),
\end{aligned}
$$

Consider

$$
\begin{aligned}
A_1^\circ &= A - \mathbb{E}_1[A] \\
&= -\frac{1}{\sqrt{b_n}}w_{11} + \frac{1}{b_n}\sum_{\substack{i\neq j\\ i,j\in I_1}} G_{ij}^{(1)}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1}\left(G_{ii}^{(1)}w_{1i}^2 - G_{ii}^{(1)}\right).
\end{aligned} \tag{48}
$$

So we have

$$
A^\circ - A_1^\circ = \frac{1}{b_n}\sum_{i\in I_1}\left(G_{ii}^{(1)} - \mathbb{E}\left[G_{ii}^{(1)}\right]\right) =: \frac{1}{b_n}\widetilde{\gamma_{n-1}}. \tag{49}
$$

Hence

$$\mathbb{E}[|A^\circ|^2] = \mathbb{E}[|A_1^\circ + b_n^{-1}\widetilde{\gamma_{n-1}}|^2] \le 2\left[\mathbb{E}[|A_1^\circ|^2] + \frac{1}{b_n^2}\mathbb{E}[|\widetilde{\gamma_{n-1}}|^2]\right].$$

From (31), we know that $\mathbb{E}[|A_1^\circ|^2] = O\left(\frac{1}{b_n}\right)$ and from (47), We have $\mathbb{E}[|\widetilde{\gamma_{n-1}}|^2] = O(1)$. Combining these two facts and using (17), we have

$$\mathrm{Var}(G_{11}(z)) = \mathbb{E}\left|\frac{1}{A} - \mathbb{E}\frac{1}{A}\right|^2 \le \mathbb{E}\left|\frac{1}{A} - \frac{1}{\mathbb{E}A}\right|^2 = \mathbb{E}\left|\frac{A^\circ}{A\mathbb{E}A}\right|^2 = O\left(\frac{1}{b_n}\right).$$

This completes the proof of second part.

Proof of (ii): *Proof of* (37)*:* Recall from (48)

$$A_1^\circ = -\frac{w_{11}}{\sqrt{b_n}} + \frac{1}{b_n}\sum_{i\neq j\in I_1} G_{ij}^{(1)} w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1} G_{ii}^{(1)}(w_{1i}^2)^\circ =: T_1 + T_2 + T_3. \tag{50}$$

We have $\mathbb{E}[|T_1|^4] = O\left(\frac{1}{b_n^2}\right)$. Now

$$\mathbb{E}\left[|T_2|^4\right] = \frac{1}{b_n^4}\mathbb{E}\left[\sum_{i\neq j,k\neq l,p\neq q,s\neq t\in I_1} G_{ij}^{(1)}\overline{G_{kl}^{(1)}}G_{pq}^{(1)}\overline{G_{st}^{(1)}} w_{1i}w_{1j}w_{1k}w_{1l}w_{1p}w_{1q}w_{1s}w_{1t}\right].$$

We use the similar technique as the moment method in the proof of the Semicircle Law. In the above sum of expectations, we have nonzero terms if the indices of w_{1m}'s match in a certain way. Non zero contribution to $\mathbb{E}[|T_2|^4]$ come from the two types of matches.

Figure 1: Type I matching

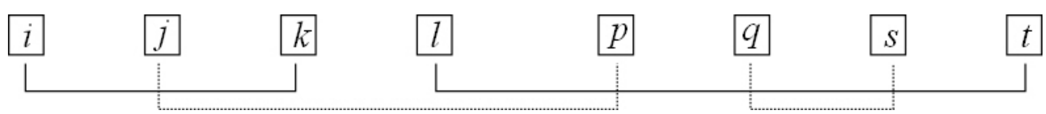


Figure 2: Type II matching

Type I: Contribution from this kind of matching is

$$\begin{aligned}
\frac{1}{b_n^4}\mathbb{E}\left[\sum_{i\neq j,p\neq k\,\in I_1} |G_{ij}^{(1)}|^2|G_{pq}^{(1)}|^2 w_{1i}^2w_{1j}^2w_{1p}^2w_{1q}^2\right] &= \frac{1}{b_n^4}\mathbb{E}\mathbb{E}_1\left[\sum_{i\neq j,p\neq k\,\in I_1} |G_{ij}^{(1)}|^2|G_{pq}^{(1)}|^2 w_{1i}^2w_{1j}^2w_{1p}^2w_{1q}^2\right]\\
&= \frac{1}{b_n^4}\sum_{i\neq j}\sum_{p\neq q}\mathbb{E}\left[|G_{ij}^{(1)}|^2|G_{pq}^{(1)}|^2\right]\\
&\le \frac{1}{b_n^4}\sum_{i\neq j}\sum_{p\neq q}\sqrt{\mathbb{E}\left[\left|G_{ij}^{(1)}\right|^4\right]\mathbb{E}\left[\left|G_{pq}^{(1)}\right|^4\right]}\\
&= \frac{1}{b_n^4}\sum_{i\neq j}\sum_{p\neq q} O\left(\frac{1}{b_n^2}\right) \quad \text{(using (33))}
\end{aligned}$$

$$= \quad O\left(\frac{1}{b_n^2}\right).$$

Type II: Similarly, contribution from the type II matching is

$$\frac{1}{b_n^4}\mathbb{E}\left[\sum_{i\neq j}\sum_{q\neq l\in I_1} G_{ij}^{(1)}\overline{G_{il}^{(1)}}G_{jq}^{(1)}\overline{G_{ql}^{(1)}}w_{1i}^2w_{1j}^2w_{1q}^2w_{1l}^2\right] = O\left(\frac{1}{b_n^2}\right).$$

Similarly, $\mathbb{E}[|T_3|^4] = O\left(\frac{1}{b_n^2}\right)$. Hence

$$\mathbb{E}[|A_1^\circ|^4] = O\left(\frac{1}{b_n^2}\right). \tag{51}$$

Using Lemma 4.4.3. from [**?**] with the help of the Poincaré inequality, we have $\mathbb{E}\left[|\widetilde{\gamma_{n-1}}|^4\right] \leq C\|\|\nabla\widetilde{\gamma_{n-1}}\|_2\|_\infty^4$, where C is a constant depends only on the constant m of the Poincaré inequality. Following the arguments given at the right side of (46) onward and (47), one can show that $\|\nabla\widetilde{\gamma_{n-1}}\|_2 \leq \frac{C}{|\Im z|^4}$, where C depends only on m. Hence $\mathbb{E}\left[|\widetilde{\gamma_{n-1}}|^4\right] = O(1)$. Consequently, using relation (49) and (51), we have $\mathbb{E}[|A^\circ|^4] = O\left(\frac{1}{b_n^2}\right)$. Then $\mathbb{E}[|A^\circ|^3] \leq \left(\mathbb{E}[|A^\circ|^4]\right)^{3/4} = O\left(\frac{1}{b_n^{3/2}}\right)$.

Proof of (38)*:* First we write B as

$$B = \left\langle G^{(1)}G^{(1)}m^{(1)}, m^{(1)}\right\rangle = \left\langle H^{(1)}m^{(1)}, m^{(1)}\right\rangle = \frac{1}{b_n}\sum_{i,j\in I_1} H_{ij}^{(1)}w_{1i}w_{1j},$$

where $H^{(1)} = G^{(1)}G^{(1)}$. Define

$$B_1^\circ := \frac{1}{b_n}\sum_{i\neq j\in I_1} H_{ij}^{(1)}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1} H_{ii}^{(1)}(w_{1i}^2)^\circ.$$

Then we can write

$$\begin{aligned}
B^\circ &= B - \mathbb{E}[B] \\
&= \frac{1}{b_n}\sum_{i\neq j\in I_1} H_{ij}w_{1i}w_{1j} + \frac{1}{b_n}\sum_{i\in I_1}\left[H_{ii}^{(1)}w_{1i}^2 - \mathbb{E}[H_{ii}^{(1)}]\right] \\
&= B_1^\circ + \frac{1}{b_n}\sum_{i\in I_1}\left(H_{ii}^{(1)} - \mathbb{E}[H_{ii}^{(1)}]\right) \\
&= B_1^\circ + \frac{1}{b_n}\overline{\gamma_{n-1}},
\end{aligned}$$

where

$$\overline{\gamma_{n-1}}(z) = \sum_{i\in I_1}\left(H_{ii}^{(1)} - \mathbb{E}[H_{ii}^{(1)}]\right) = \sum_{i\in I_1}\sum_{j=2}^{n}\left(G_{ij}^{(1)}G_{ji}^{(1)} - \mathbb{E}\left[G_{ij}^{(1)}G_{ji}^{(1)}\right]\right) = \frac{d}{dz}\widetilde{\gamma_{n-1}}(z).$$

Proceeding as in the estimate of $\mathbb{E}[|A_1^\circ|^4]$, we can show

$$\mathbb{E}[|B_1^\circ|^4] = O\left(\frac{1}{b_n^2}\right). \tag{52}$$

We have shown that $\mathbb{E}[|\widetilde{\gamma_{n-1}}(z)|^4] = O(1)$. Using this fact and Cauchy's theorem we have $\mathbb{E}[|\overline{\gamma_{n-1}}(z)|^4] = O(1)$. Hence we have the result.

Proof of (iii):

$\mathrm{Var}\left\{b_n \mathbb{E}_1\left[A^\circ(z_1)A^\circ(z_2)\right]\right\} = \mathrm{Var}(T_1) + \mathrm{Var}(T_2) + \mathrm{Var}(T_3) + 2\mathrm{Cov}(T_1, T_2) + 2\mathrm{Cov}(T_2, T_3) + 2\mathrm{Cov}(T_3, T_1),$

where

$$T_1 = \frac{2}{b_n}\sum_{i,j\in I_1} G_{ij}^{(1)}(z_1)G_{ij}^{(1)}(z_2),\ T_2 = \frac{\kappa_4}{b_n}\sum_{i\in I_1} G_{ii}^{(1)}(z_1)G_{ii}^{(1)}(z_2) \text{ and } T_3 = \frac{1}{b_n}\widetilde{\gamma_{n-1}}(z_1)\widetilde{\gamma_{n-1}}(z_2).$$

Now, $\mathrm{Var}(T_2) = \frac{\kappa_4^2}{b_n^2}\mathrm{Var}\left\{\sum_{i\in I_1} G_{ii}^{(1)}(z_1)G_{ii}^{(1)}(z_2)\right\}$ and

$$\begin{aligned}
\mathrm{Var}\left\{G_{ii}^{(1)}(z_1)G_{ii}^{(1)}(z_2)\right\} &= \mathbb{E}\left|G_{ii}^{(1)}(z_1)G_{ii}^{(1)}(z_2) - \mathbb{E}[G_{ii}^{(1)}(z_1)G_{ii}^{(1)}(z_2)]\right|^2 \\
&\leq \frac{2}{|\Im z_1|^2}\mathrm{Var}\left(G_{ii}^{(1)}(z_2)\right) + \frac{2}{|\Im z_2|^2}\mathrm{Var}\left(G_{ii}^{(1)}(z_1)\right) \\
&= \left(\frac{1}{|\Im z_1|^2} + \frac{1}{|\Im z_2|^2}\right) O\left(\frac{1}{b_n}\right).
\end{aligned}$$

Therefore,

$$\mathrm{Var}(T_2) \leq \frac{\kappa_4^2}{b_n^2}\left(b_n O\left(\frac{1}{b_n}\right) + b_n^2 O\left(\frac{1}{b_n}\right)\right) = O\left(\frac{1}{b_n}\right).$$

Now

$$\begin{aligned}
\mathrm{Var}(T_3) &\leq \frac{1}{b_n^2}\mathrm{Var}\left(\widetilde{\gamma_{n-1}}(z_1)\widetilde{\gamma_{n-1}}(z_2)\right) \\
&\leq \frac{1}{b_n^2}\mathbb{E}\left[|\widetilde{\gamma_{n-1}}(z_1)|^2|\widetilde{\gamma_{n-1}}(z_2)|^2\right] \\
&\leq \frac{1}{b_n^2}\sqrt{\mathbb{E}\left[|\widetilde{\gamma_{n-1}}(z_1)|^4\right]}\sqrt{\mathbb{E}\left[|\widetilde{\gamma_{n-1}}(z_2)|^4\right]} \\
&= \frac{1}{b_n^2}O(1).
\end{aligned}$$

Last equality holds, since $\mathbb{E}\left[|\widetilde{\gamma_{n-1}}(z_1)|^4\right] = O(1)$. And finally

$$\mathrm{Var}(T_1) = \frac{4}{b_n^2}\mathrm{Var}\left(\sum_{i,j\in I_1} G_{ij}^{(1)}(z_1)G_{ij}^{(1)}(z_2)\right).$$

Now using the Poincaré inequality

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{i,j\in I_1} G_{ij}^{(1)}(z_1)G_{ij}^{(1)}(z_2)\right) \\
\leq\ & \frac{1}{m}\sum_{(s,t)\in I_n^+}\mathbb{E}\left[\left|\frac{\partial}{\partial w_{st}}\sum_{i,j\in I_1} G_{ij}^{(1)}(z_1)G_{ij}^{(1)}(z_2)\right|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{mb_n}\sum_{(s,t)\in I_n^+}\mathbb{E}\left[\left|\sum_{i,j\in I_1}G_{is}^{(1)}(z_1)G_{tj}^{(1)}(z_1)G_{ij}^{(1)}(z_2)+G_{ij}^{(1)}(z_1)G_{is}^{(1)}(z_2)G_{tj}^{(1)}(z_2)\right|^2\right]\\
&\leq \frac{2}{mb_n}\sum_{(s,t)\in I_n^+}\mathbb{E}\left[\left|\sum_{i,j\in I_1}G_{is}^{(1)}(z_1)G_{tj}^{(1)}(z_1)G_{ij}^{(1)}(z_2)\right|^2\right]+\frac{2}{mb_n}\sum_{(s,t)\in I_n^+}\mathbb{E}\left[\left|\sum_{i,j\in I_1}G_{ij}^{(1)}(z_1)G_{is}^{(1)}(z_2)G_{tj}^{(1)}(z_2)\right|^2\right]\\
&=: I_1+I_2.
\end{aligned}
$$

We estimate

$$
\begin{aligned}
I_1 &= \frac{2}{mb_n}\sum_{(s,t)\in I_n^+}\mathbb{E}\left[\left|\sum_{i,j\in I_1}G_{is}^{(1)}(z_1)G_{tj}^{(1)}(z_1)G_{ij}^{(1)}(z_2)\right|^2\right]\\
&= \frac{2}{mb_n}\sum_{(s,t)\in I_n^+}\mathbb{E}\left[\left|\sum_{i\in I_1}G_{is}^{(1)}(z_1)G_{ti}^{(1)}(z_1,z_2)\right|^2\right]\\
&= \frac{2}{mb_n}\sum_{(s,t)\in I_n^+}\mathbb{E}\left[\left|G_{st}^{(1)}(z_1,z_2,z_1)\right|^2\right]\\
&\leq \frac{2}{mb_n}\mathbb{E}\left[\sum_{s,t=1}^{n}\left|G_{st}^{(1)}(z_1,z_2,z_1)\right|^2\right]\\
&= \frac{2}{mb_n}\mathbb{E}\left[\|A\|_{Fb}^2\right]\\
&= \frac{2}{mb_n}\mathbb{E}\left[\sum_{i=1}^{n}\beta_i^2\right]\\
&\leq \frac{C(z_1,z_2)}{mb_n}O(b_n)=O(1),
\end{aligned}
$$

where $\|\cdot\|_{Fb}$ is the Frobenius norm, β_i are the eigenvalues of VV^*, and V is the following matrix

$$
\begin{aligned}
V_{n\times n} &= \begin{bmatrix} G_{11}^{(1)}(z_1) & G_{12}^{(1)}(z_1) & \cdots & G_{1k_n}^{(1)}(z_1)\\ G_{21}^{(1)}(z_1) & G_{22}^{(1)}(z_1) & \cdots & G_{2k_n}^{(1)}(z_1)\\ \vdots & \vdots & & \vdots\\ G_{n1}^{(1)}(z_1) & G_{n2}^{(1)}(z_1) & \cdots & G_{nk_n}^{(1)}(z_1)\end{bmatrix}_{n\times k_n}\begin{bmatrix} G_{11}^{(1)}(z_2) & G_{12}^{(1)}(z_2) & \cdots & G_{1k_n}^{(1)}(z_2)\\ G_{21}^{(1)}(z_2) & G_{22}^{(1)}(z_2) & \cdots & G_{2k_n}^{(1)}(z_2)\\ \vdots & \vdots & & \vdots\\ G_{k_n1}^{(1)}(z_2) & G_{k_n2}^{(1)}(z_2) & \cdots & G_{k_nk_n}^{(1)}(z_2)\end{bmatrix}_{k_n\times k_n}\\
&\times\begin{bmatrix} G_{11}^{(1)}(z_1) & G_{12}^{(1)}(z_1) & \cdots & G_{1n}^{(1)}(z_1)\\ G_{21}^{(1)}(z_1) & G_{22}^{(1)}(z_1) & \cdots & G_{2n}^{(1)}(z_1)\\ \vdots & \vdots & & \vdots\\ G_{k_n1}^{(1)}(z_1) & G_{t2}^{(1)}(z_1) & \cdots & G_{k_nn}^{(1)}(z_1)\end{bmatrix}_{k_n\times n}.
\end{aligned}
$$

Here we denoted the elements of set I_1 as $I_1=\{1,2,\ldots,k_n\}$. Observe that $k_n=2b_n$. Rank of $V\leq k_n=O(b_n)$. This implies

$$
\sum_{i=1}^{n}\beta_i^2\leq k_nC(z_1,z_2)=O(b_n)C(z_1,z_2).
$$

Therefore, $\mathrm{Var}(T_1)=O\left(\frac{1}{b_n^2}\right)$, and hence $\mathrm{Var}\{b_n\mathbb{E}_1\left[A^\circ(z_1)A^\circ(z_2)\right]\}=O\left(\frac{1}{b_n}\right)$.

Second part of **(iii)** follows from the following two facts with the help of Cauchy's theorem.

$$b_n \mathbb{E}_1 \left[A^\circ(z_1)B^\circ(z_2)\right] = b_n \frac{d}{dz_2}\mathbb{E}_1 \left[A^\circ(z_1)A^\circ(z_2)\right]$$

$$\text{and} \qquad \operatorname{Var}\left\{b_n \mathbb{E}_1 \left\{A^\circ(z_1)A^\circ(z_2)\right\}\right\} = O\left(\frac{1}{b_n}\right).$$

Here we skip the details.
Proof of (iv): Using (25) and (30), and proceeding as the proof of proposition 2,

$$\begin{aligned}
\mathbb{E}\left[\left|\gamma_{n-1}^\circ(z) - \gamma_n^\circ(z)\right|^4\right] &= \mathbb{E}\left[\left|\left(\mathrm{Tr}G^{(1)}(z) - \mathbb{E}[\mathrm{Tr}G^{(1)}(z)]\right) - \left(\mathrm{Tr}G(z) - \mathbb{E}[\mathrm{Tr}G(z)]\right)\right|^4\right] \\
&= \mathbb{E}\left[\left|\frac{1+B(z)}{A(z)} - \mathbb{E}\left[\frac{1+B(z)}{A(z)}\right]\right|^4\right] \\
&\leq \frac{C}{|\Im z|^8}\left[\mathbb{E}\left[|A^\circ|^4\right] + \mathbb{E}\left[|B^\circ|^4\right] + \mathbb{E}\left[|A^\circ|^4\right]\right] = O\left(\frac{1}{b_n^2}\right).
\end{aligned}$$

The last equality follows from the estimates (37) and (38).

Using martingale differences as in the proof of Proposition 2,

$$\mathbb{E}\left[|\gamma_n^\circ|^4\right] \leq Cn\sum_{k=1}^{n}\mathbb{E}\left[|\gamma_n - \mathbb{E}_k[\gamma_n]|^4\right].$$

Consider for $k=1$, others will be similar.

$$\begin{aligned}
\mathbb{E}\left[|\gamma_n - \mathbb{E}_1[\gamma_n]|^4\right] &= \mathbb{E}\left[\left|\mathrm{Tr}(G - G^{(1)}) - \mathbb{E}_1\left[\mathrm{Tr}(G - G^{(1)})\right]\right|^4\right] \\
&= \mathbb{E}\left[\left|\frac{1+B(z)}{A(z)} - \mathbb{E}_1\left[\frac{1+B(z)}{A(z)}\right]\right|^4\right] \\
&\leq C_1(z)\mathbb{E}[|A_1^\circ|^4] + C_2(z)\mathbb{E}[|B_1^\circ|^4] \\
&= O\left(\frac{1}{b_n^2}\right).
\end{aligned}$$

The last equality follows from (51) and (52). Hence we have the result.

Proof of (v): Using resolvent identity,

$$(X_2 - zI)^{-1} = (X_1 - zI)^{-1} + (X_1 - zI)^{-1}(X_1 - X_2)(X_2 - zI)^{-1},$$

we have

$$zG_{11}(z) = -1 + \sum_{(1,k)\in I_n} m_{1k}G_{k1}, \tag{53}$$

where I_n is defined in (1) and m_{ij}s are defined in (2). Now to analyse the terms $\mathbb{E}[m_{1k}G_{k1}]$, we use the following (see eg. [**?**]): Given ξ, a real valued random variable with $p+2$ finite moments, and ϕ, a function from $\mathbb{C} \to \mathbb{R}$ with $p+1$ continuous and bounded derivatives then:

$$\mathbb{E}[\xi\phi(\xi)] = \sum_{a=0}^{p}\frac{\kappa_{a+1}}{a!}\mathbb{E}\left[\phi^{(a)}(\xi)\right] + \epsilon_{p+1} \tag{54}$$

where κ_a is the a-th cumulant of ξ, $|\epsilon_{p+1}| \le C \sup_t |\phi^{(p+1)}(t)| \mathbb{E}[|\xi|^{p+2}]$ and C depends only on p. Since $f_n(z) = \frac{1}{n}\mathbb{E}[\mathrm{Tr}G(z)] = \mathbb{E}[G_{11}(z)]$, using (53) and (54) we get

$$zf_n(z) = -1 + \sum_{(1,k)\in I_n} \mathbb{E}[m_{1k}G_{k1}] = -1 - \sum_{k\in I_1} \frac{1}{b_n}\mathbb{E}\left[G_{k1}^2 + G_{kk}G_{11}\right] + r_n, \tag{55}$$

where r_n contains the third cumulant term corresponding to $p=2$ in (54) for $k \neq 1$, and the error terms due to the truncation of the decoupling formula (54) at $p=2$ for $k \neq 1$ and at $p=0$ for $k=1$. We write (55)

$$\begin{aligned} zf_n(z) &= -1 - \frac{1}{b_n}\mathbb{E}[G_{11}]\mathbb{E}\left[\sum_{k\in I_1} G_{kk}\right] - \frac{1}{b_n}\mathrm{Cov}\left(G_{11}, \sum_{k\in I_1} G_{kk}\right) - \frac{1}{b_n}\mathbb{E}\left[\sum_{k\in I_1} G_{k1}^2\right] + r_n \\ &= -1 - f_n(z)(2f_n(z)) - \frac{1}{b_n}\mathrm{Cov}\left(G_{11}, \sum_{k\in I_1} G_{kk}\right) - \frac{1}{b_n}\mathbb{E}\left[\sum_{k\in I_1} G_{k1}^2\right] + r_n. \end{aligned}$$

Now, by the Cauchy-Schwarz inequality and (47) we get

$$\begin{aligned} \frac{1}{b_n}\left|\mathrm{Cov}\left(G_{11}, \sum_{k\in I_1} G_{kk}\right)\right| &\le \frac{1}{b_n}\sqrt{\mathrm{Var}(G_{11})}\sqrt{\mathrm{Var}\left(\sum_{k\in I_1} G_{kk}\right)} \\ &\le \frac{1}{b_n} 2|\Im z|^{-1}\sqrt{O(|\Im z|^{-4})} \\ &= O\left(\frac{1}{b_n|\Im z|^3}\right). \end{aligned}$$

Also notice that

$$\frac{1}{b_n}\left|\mathbb{E}\left(\sum_{k\in I_1} G_{k1}^2\right)\right| \le \frac{1}{b_n}|\Im z|^{-2}.$$

We claim $r_n = O\left(\frac{1}{b_n|\Im z|^4}\right)$. To prove this, observe that the third cumulant term gives

$$\frac{\kappa_3}{2b_n^{3/2}}\mathbb{E}\left[\sum_{k\in I_1} 2(G_{1k})^3 + 6G_{11}G_{1k}G_{kk}\right] \tag{56}$$

Since

$$\sum_{k\in I_1} |G_{1k}|^2 \le \|G\|^2 \le |\Im z|^{-2} \quad \text{and} \quad |G_{ij}| \le |\Im z|^{-1},$$

we conclude that the third cumulant term contributes $O\left(\frac{1}{b_n|\Im z|^3}\right)$ to r_n. In a similar manner, the error due to truncation of decoupling formula (54) at $p=2$ is $O\left(\frac{1}{b_n|\Im z|^4}\right)$. Similarly, the error term due to truncation of decoupling formula at $p=0$ for $k=1$ is $O\left(\frac{1}{b_n|\Im z|^2}\right)$. Thus the claim is proved. Hence

$$zf_n(z) = -1 - 2f_n^2(z) + O\left(\frac{1}{b_n|\Im z|^4}\right) \quad \text{for } z \in \mathbb{C}\backslash\mathbb{R}.$$

Now following similar argument given in the proof of (3.1) in [**?**], one can show that

$$|f_n(z) - f(z)| \le O\left(\frac{1}{b_n|\Im z|^6}\right)$$

where $f(z) = \frac{1}{4}(-z + \sqrt{z^2 - 8})$.

Proof of (vi): Recall $A(z) = z - b_n^{-1/2} w_{11} + b_n^{-1} \sum_{i,j \in I_1} G_{ij}^{(1)} w_{1i} w_{1j}$. Now using (41) with G replaced by $G^{(1)}$, we have

$$(\mathbb{E}[A(z)])^{-1} = \frac{1}{z + b_n^{-1} \sum_{j \in I_1} \mathbb{E}[G_{jj}^{(1)}]} = \frac{1}{z + 2f_n(z)} = (z + 2f(z))^{-1} + O(b_n^{-1}) = -f(z) + O(b_n^{-1})$$

Hence $(\mathbb{E}[A(z)])^{-1} = -f(z) + O(b_n^{-1})$. To prove the second part, observe that

$$\mathbb{E}[B(z)] = \frac{1}{b_n} \mathbb{E}\left[\sum_{i,j \in I_1} (G^{(1)} G^{(1)})_{ij} w_{1i} w_{1j}\right] = \frac{1}{b_n} \mathbb{E}\left[\sum_{i \in I_1} (G^{(1)} G^{(1)})_{ii}\right] = \frac{1}{b_n} \mathbb{E}\left[\sum_{i \in I_1} \sum_{k=2}^{n} G_{ik}^{(1)} G_{ki}^{(1)}\right] = \frac{1}{b_n} \sum_{i \in I_1} \frac{d}{dz} G_{ii}^{(1)}$$

Again using (41) and Cauchy's integral formula, we have

$$\mathbb{E}[B(z)] = \frac{d}{dz}(2f_n(z)) = 2f'(z) + O(b_n^{-1}).$$

This completes the proof of Lemma 2. □

4.1 Proof of (29):

Proof. We have to find the limit of

$$\mathbb{E}[T_n] = \frac{2}{b_n} \mathbb{E}\left[\sum_{i,j \in I_1} G_{ij}^{(1)}(z) G_{ij}^{(1)}(z_\mu)\right]$$

as $n \to \infty$, where $I_1 = \{2 \le i \le n : (1, i) \in I_n\}$. Let $f, g \in C_b(\mathbb{R})$. Define a bilinear form on $C_b(\mathbb{R})$ as

$$\langle f, g \rangle_n = \frac{1}{b_n} \mathbb{E}\left[\sum_{i,j \in I_1} f(M)_{ij} g(M)_{ji}\right]. \tag{57}$$

Then $\mathbb{E}[T_n] = \langle h(M), h_\mu(M) \rangle_n$, where $h(x) = (x - z)^{-1}$ and $h_\mu(x) = (x - z_\mu)^{-1}$.

Lemma 3. *For $f, g \in C_b(\mathbb{R})$ the limit $\langle f, g \rangle = \lim_{n \to \infty} \langle f, g \rangle_n$ exists.*

Proof. The idea of the proof is similar to the proof of Lemma 3.11 of [?]. First we prove this result for monomials. Although monomials are unbounded, still (57) makes sense for all n, since all moments of the entries of M are finite. Consider $f(x) = x^l$ and $g(x) = x^m$ where $l, m \in \mathbb{N}$. Then

$$\langle x^l, x^m \rangle_n = \frac{1}{b_n^{1+(l+m)/2}} \sum_{\substack{(i_0, i_1), (i_1, i_2), \ldots, (i_{l+m-1}, i_0) \in I_n \\ i_o, i_l \in I_1}} \mathbb{E}\left[w_{i_0 i_1} w_{i_1 i_2} \ldots w_{i_{l+m-1} i_0}\right]$$

If $(l+m)$ is odd then $\langle x^l, x^m \rangle_n \to 0$ using independence of matrix entries and $\mathbb{E}(w_{ij}) = 0$, and order counting of independent vertices. The argument is similar to the combinatorial argument given in the proof of Wigner semicircular law (see [?]). We leave it for the reader.

Now we assume $l + m$ is even. Then

$$\langle x^l, x^m \rangle_n = \frac{1}{b_n^{1+(l+m)/2}} \sum_{\substack{(i_0, i_1), (i_1, i_2), \ldots, (i_{l+m-1}, i_0) \in I_n \\ i_o, i_l \in I_1}} \mathbb{E}\left[w_{i_0 i_1} w_{i_1 i_2} \ldots w_{i_{l+m-1} i_0}\right]$$

$$= \frac{1}{b_n^{1+(l+m)/2}} \sum_{\substack{(i_0,i_1),(i_1,i_2),\ldots,(i_{l+m-1},i_0)\in I_n \\ i_o,i_l\in I_1}} \mathbb{E}\left[w_{1i_0}w_{i_0i_1}w_{i_1i_2}\ldots w_{i_{l+m-1}i_0}w_{i_01}\right] + O(b_n^{-1})$$

$$= \frac{1}{b_n^{1+(l+m)/2}} \sum_{\substack{(i_0,i_1),(i_1,i_2),\ldots,(i_{l+m-1},i_0)\in I_n \\ (1,i_o),(1,i_l)\in I_n}} \mathbb{E}\left[w_{1i_0}w_{i_0i_1}w_{i_1i_2}\ldots w_{i_{l+m-1}i_0}w_{i_01}\right] + O(b_n^{-1}) \quad (58)$$

The second last equality in (58) holds due to order calculation of independent vertices and independence of matrix entries. Now define for $k = 1, 2, \ldots, l+m$,

$$x_k = \begin{cases} i_k - i_{k-1} & \text{if} \quad |i_k - i_{k-1}| \le b_n \\ (i_k - i_{k-1}) - n & \text{if} \quad i_k - i_{k-1} > b_n \\ n + (i_k - i_{k-1}) & \text{if} \quad i_k - i_{k-1} < -b_n \end{cases} \quad \text{with } i_{l+m} = i_0, \text{ and}$$

$$x_0 = \begin{cases} i_0 - 1 & \text{if} \quad |i_0 - 1| \le b_n \\ (i_0 - 1) - n & \text{if} \quad i_0 - 1 > b_n \end{cases} \quad \text{and} \quad x_{l+m+1} = \begin{cases} 1 - i_0 & \text{if} \quad |1 - i_0| \le b_n \\ n + (1 - i_0) & \text{if} \quad 1 - i_0 < -b_n. \end{cases}$$

Note, $x_0 = -x_{l+m+1}$. Since l, m are fixed and $b_n \to \infty$, for large n the restrictions $\{(i_0,i_1),(i_1,i_2),\ldots,(i_{l+m-1},i_0) \in I_n \text{ and } (1,i_0),(1,i_l) \in I_n\}$ are equivalent to $\{|x_0|,|x_1|,\ldots,|x_{l+m}| \le b_n,\ x_0 + x_1 + \cdots + x_{l+m} + x_{l+m+1} = 0 \text{ and } |x_0+x_1+\cdots+x_l| \le b_n\}$. Also observe that $x_0+x_1+\cdots+x_{l+m}+x_{l+m+1} = 0$ is same as $x_1+\cdots+x_{l+m} = 0$ since $x_0 = -x_{l+m+1}$. Therefore for large n

$$\langle x^l, x^m\rangle_n = \frac{1}{b_n^{1+(l+m)/2}} \sum_{\substack{x_1+\cdots+x_{l+m}=0 \\ |x_i|\le b_n, 0\le i\le l+m,\ |x_0+x_1+\cdots+x_l|\le b_n}} \mathbb{E}\left[w_{1i_0}w_{i_0i_1}w_{i_1i_2}\ldots w_{i_{l+m-1}i_0}w_{i_01}\right) + O(b_n^{-1}].$$

Without loss of generality, we assume that $l \le m$. Each $\{i_0, i_1, i_2, \ldots, i_{l+m-1}, i_0\}$ is a closed path such that distance between the end points of each edge is bounded by b_n. As in the proof of Wigner semicircular law only the paths whose edges are pair matched contributes to the limit, here also, only such paths contribute to the limit. And contribution of each path is $\mathbb{E}(w_{1i_0}w_{i_0i_1}\ldots w_{i_{l+m-1}i_0}w_{i_01}) = 1$ since $\mathbb{E}(w_{ij}^2) = 1$. Each such path corresponds to a Dyck path of length $(l+m)$. Recall that a Dyck path $(S(0), S(1), \ldots, S(l+m))$ of length $(l+m)$ satisfies (see [**?**])

$$S(0) = S(l+m) = 0,\ S(1), S(2), \ldots, S(l+m-1) \ge 0 \text{ and } |S(i+1) - S(i)| = 1, \text{ for } i = 0, 1, \ldots, l+m-1.$$

Specifically, $S(t+1) - S(t) = 1$ if the non-oriented edge (i_t, i_{t+1}) appears in $\{i_0, i_1, \ldots, i_{l+m-1}, i_0\}$ for the first time and $S(t+1) - S(t) = -1$ if the edge (i_t, i_{t+1}) appears in $\{i_0, i_1, \ldots, i_{l+m-1}, i_0\}$ for the second time.

Here each Dyck path does not give equal contribution to the limit due to the condition that $(1, i_l) \in I_n$ and in terms of x_i, which is same as $|x_0 + x_1 + \cdots + x_l| \le b_n$. We have to take into account this condition. Suppose $S(l) = k,\ 0 \le k \le l$. Then during the first l steps of the path $\{i_0, i_1, \ldots, i_{l+m-1}, i_0\}$, k edges appear only once and $(l-k)/2$ edges appear twice. The edges appearing twice, the corresponding two number x_i have same absolute value but with different sign. We rename the remaining k numbers x_i which appear only once as $y_1, y_2, \ldots, y_k$ (according to their order of appearance) and x_0 as y_0. So the condition $|x_0 + x_1 + \ldots + x_l| \le b_n$ reduces to $|y_0 + y_1 + \ldots + y_k| \le b_n$. Therefore

$$\begin{aligned}\langle x^l, x^m\rangle_n &= \frac{1}{b_n^{1+(l+m)/2}} \sum_{k=0}^{l} \#\{\text{Dyck path of length } l+m \text{ with } S(l) = k\} \\ &\quad \times \#\{|y_0| \le b_n, |y_1| \le b_n, \ldots, |y_k| \le b_n, \ldots, |y_{l+m}| \le b_n, |y_0 + y_1 + \cdots + y_k| \le b_n\} + O(b_n^{-1}).\end{aligned}$$

and

$$\langle x^l, x^m\rangle = \lim_{n\to\infty} \langle x^l, x^m\rangle_n$$

$$
\begin{aligned}
&= (\sqrt{2})^{l+m+2}\sum_{k=0}^{l}\#\{\text{Dyck path of length } l+m \text{ with } S(l)=k\} \\
&\quad \times \mathrm{Vol}\{|t_0|\le 1/2, |t_1|\le 1/2, \ldots, |t_{\frac{l+m}{2}}|\le 1/2, |t_0+t_1+\cdots+t_k|\le 1/2\} \\
&= (\sqrt{2})^{l+m+2}\sum_{k=0}^{l}\#\{\text{Dyck path of length } l+m \text{ with } S(l)=k\}\times P(|T_0+T_1+\cdots+T_k|\le 1/2)
\end{aligned}
$$

where $T_0, T_1, \ldots, T_{\frac{l+m}{2}}$ are independent random variables uniformly distributed on $[-1/2, 1/2]$. Let $S_{k+1} = T_0 + T_1 + \ldots + T_k$. Then

$$\mathbb{E}\left[e^{ixS_{k+1}}\right] = \left(\mathbb{E}[e^{ixT_0}]\right)^{k+1} = \left(\frac{\sin x/2}{x/2}\right)^{k+1}.$$

Using inversion formula, the density of S_{k+1} is given by

$$f_{k+1}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixs}\left(\frac{\sin x/2}{x/2}\right)^{k+1} dx.$$

Now

$$\gamma_{k+1} := P(|S_{k+1}|\le 1/2) = \int_{-1/2}^{1/2} f_{k+1}(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x/2}\right)^{k+2} dx = f_{k+2}(0),$$

using [?] we get exact formula of γ_{k+1}:

$$\gamma_{k+1} = \begin{cases} \frac{1}{(k+1)!}\sum_{s=0}^{(k+1)/2}(-1)^s\binom{k+2}{s}\left(\frac{k+1}{2}-s+\frac{1}{2}\right)^{k+1} & \text{if } \ k+1 \text{ even} \\ \frac{1}{(k+1)!}\sum_{s=0}^{k/2}(-1)^s\binom{k+2}{s}\left(\frac{k+1}{2}-s+\frac{1}{2}\right)^{k+1} & \text{if } \ k+1 \text{ odd.} \end{cases} \tag{59}$$

The number of Dyck path of length $l+m$ with $S(l)=k$ is

$$\left[\binom{l}{\frac{l-k}{2}} - \binom{l}{\frac{l-k-2}{2}}\right]\times\left[\binom{m}{\frac{m-k}{2}} - \binom{m}{\frac{m-k-2}{2}}\right] = \frac{(k+1)^2}{(l+1)(m+1)}\binom{l+1}{\frac{l+k+2}{2}}\binom{m+1}{\frac{m+k+2}{2}}. \tag{60}$$

Hence from (59) and (60), we get

$$\langle x^l, x^m\rangle = (\sqrt{2})^{l+m+2}C_{l,m}$$

where $C_{l,m}=0$ if $(l+m)$ is odd and

$$
\begin{aligned}
C_{l,m} &= \sum_{k=0}^{l}\frac{(k+1)^2}{(l+1)(m+1)}\binom{l+1}{\frac{l+k+2}{2}}\binom{m+1}{\frac{m+k+2}{2}}\gamma_{k+1} \\
&= \begin{cases} \sum_{k=0}^{l/2}\frac{(2k+1)^2}{(l+1)(m+1)}\binom{l+1}{\frac{l-2k}{2}}\binom{m+1}{\frac{m-2k}{2}}\gamma_{2k+1} & \text{if } \ l \text{ even} \\ \sum_{k=0}^{(l-1)/2}\frac{(2k+2)^2}{(l+1)(m+1)}\binom{l+1}{\frac{l-2k-1}{2}}\binom{m+1}{\frac{m-2k-1}{2}}\gamma_{2k+2} & \text{if } \ l \text{ odd} \end{cases}
\end{aligned}
$$

if $(l+m)$ is even and $l\le m$, otherwise, $C_{l,m}=C_{m,l}$. If f,g are polynomials, $f(x)=\sum_{i=0}^{p}a_ix^i$, $g(x)=\sum_{i=0}^{q}b_ix^i$, then by linearity

$$\langle f,g\rangle = \sum_{i=0}^{p}\sum_{j=0}^{q}a_ib_j(\sqrt{2})^{i+j+2}C_{i,j}. \tag{61}$$

For general bounded continuous functions f,g, to show that $\langle f,g\rangle$ exists we have to use the Stone-Weierstrass theorem to approximate f,g by appropriate polynomial and then (61). The argument is similar to the argument given in the proof of Lemma 3.11 of [?]. We skip the details. $\square$

In the next lemma we diagonalize the bilinear form $\langle f, g\rangle$.

Lemma 4. *Let $\{U_n(x)\}_{n\geq 0}$ be the rescaled Chebyshev polynomial of the second kind on $[-2\sqrt{2}, 2\sqrt{2}]$,*

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \left(\frac{x}{\sqrt{2}}\right)^{n-2k}.$$

Then $\{U_n(x)\}$ are orthogonal with respect to the bilinear form (61), that is,

$$\langle U_n, U_m\rangle = 2\delta_{nm}\gamma_{n+1}, \tag{62}$$

where γ_{n+1} is defined in (59).

Proof. The proof of this lemma is similar to the proof of Lemma 3.12 of [**?**]. For sake of completeness we outline it here. Since $\langle x^l, x^m\rangle = 0$ if $l+m$ is odd, from linearity $\langle U_l, U_m\rangle = 0$ if $l+m$ is odd. We are left to compute $\langle U_{2n}, U_{2m}\rangle$ and $\langle U_{2n+1}, U_{2m+1}\rangle$. We first compute $\langle x^{2l}, U_{2n}\rangle$ and $\langle x^{2l+1}, U_{2n+1}\rangle$ for $l = 0, 1, \ldots, n$.

$$\begin{aligned}
\langle x^{2l}, U_{2n}\rangle &= (\sqrt{2})^{2l+2}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}C_{2l,2n-2k} \\
&= (\sqrt{2})^{2l+2}\left[\sum_{k=0}^{n-l}(-1)^k\binom{2n-k}{k}\sum_{t=0}^{l}\frac{(2t+1)^2}{(2l+1)(2n-2k+1)}\binom{2l+1}{l-t}\binom{2n-2k+1}{n-k-t}\gamma_{2t+1}\right. \\
&\quad \left. + \sum_{k=n-l+1}^{n}(-1)^k\binom{2n-k}{k}\sum_{t=0}^{n-k}\frac{(2t+1)^2}{(2l+1)(2n-2k+1)}\binom{2l+1}{l-t}\binom{2n-2k+1}{n-k-t}\gamma_{2t+1}\right] \\
&= (\sqrt{2})^{2l+2}\sum_{t=0}^{l}\frac{(2t+1)^2}{2l+1}\binom{2l+1}{l-t}\left[\sum_{k=0}^{n-t}\frac{(-1)^k(2n-k)!}{k!(n-k-t)!(n-k+t+1)!}\right]\gamma_{2t+1} \\
&= (\sqrt{2})^{2l+2}\sum_{t=0}^{l}\frac{(2t+1)^2}{2l+1}\binom{2l+1}{l-t}G_1(n,t)\gamma_{2t+1},
\end{aligned}$$

where

$$G_1(n,t) = \sum_{k=0}^{n-t}\frac{(-1)^k(2n-k)!}{k!(n-k-t)!(n-k+t+1)!}.$$

Similarly,

$$\begin{aligned}
\langle x^{2l+1}, U_{2n+1}\rangle &= (\sqrt{2})^{2l+3}\sum_{t=0}^{l}\frac{(2t+2)^2}{2l+2}\binom{2l+2}{l-t}\left[\sum_{k=0}^{n-t}\frac{(-1)^k(2n+1-k)!}{k!(n-k-t)!(n-k+t+2)!}\right]\gamma_{2t+2} \\
&= (\sqrt{2})^{2l+3}\sum_{t=0}^{l}\frac{(2t+2)^2}{2l+2}\binom{2l+2}{l-t}G_2(n,t)\gamma_{2t+2},
\end{aligned}$$

where

$$G_2(n,t) = \sum_{k=0}^{n-t}\frac{(-1)^k(2n+1-k)!}{k!(n-k-t)!(n-k+t+2)!}.$$

$G_1(n,t)$ and $G_2(n,t)$ can be written in terms of hypergeometric function as follows:

$$G_1(n,t) = \frac{(2n)!}{(n-t)!(n+t+1)!}\ {}_2F_1\left({-(n-t), -(n+t+1) \atop -2n}\,;\ 1\right)$$

$$G_2(n,t) \quad = \quad \frac{(2n+1)!}{(n-t)!(n+t+2)!}\ {}_2F_1\left({-(n-t),-(n+t+2) \atop -2n-1};\ 1\right)$$

where ${}_2F_1$ is a hypergeometric function. By the Chu-Vandermonde identity (see [?]), we have

$$\begin{aligned} {}_2F_1\left({-(n-t),-(n+t+1) \atop -2n};\ 1\right) &= \frac{(-n+t+1)_{n-t}}{(-2n)_{n-t}}, \\ {}_2F_1\left({-(n-t),-(n+t+2) \atop -2n-1};\ 1\right) &= \frac{(-n+t+1)_{n-t}}{(-2n-1)_{n-t}}, \end{aligned}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. Since

$$(-n+t+1)_{n-t} = \begin{cases} 0 & \text{if} \quad t = 0,1,\cdots,n-1 \\ 1 & \text{if} \quad t = n \end{cases}$$

we have $G_1(n,t) = 0$, $G_2(n,t) = 0$ for $t = 0,1,\ldots,n-1$ and $G_1(n,n) = 1/(2n+1)$, $G_2(n,n) = 1/(2n+2)$. Therefore, $\langle x^{2l}, U_{2n}\rangle = 0$ for $0 \le l \le n-1$ and

$$\langle x^{2n}, U_{2n}\rangle = (\sqrt{2})^{2n+2}\gamma_{2n+1}.$$

Similarly, $\langle x^{2l+1}, U_{2n+1}\rangle = 0$ for $0 \le l \le n-1$ and

$$\langle x^{2n+1}, U_{2n+1}\rangle = (\sqrt{2})^{2n+3}\gamma_{2n+2}.$$

Therefore

$$\langle U_{2n}, U_{2n}\rangle = 2\gamma_{2n+1} \text{ and } \langle U_{2n+1}, U_{2n+1}\rangle = 2\gamma_{2n+2}.$$

This completes the proof of the lemma. □

Now we complete the proof of (29). For $f, g \in C_b(\mathbb{R})$, if

$$f_k = \frac{1}{4\pi}\int_{-2\sqrt{2}}^{2\sqrt{2}} f(x)U_k(x)\sqrt{8-x^2}dx, \quad g_k = \frac{1}{4\pi}\int_{-2\sqrt{2}}^{2\sqrt{2}} g(x)U_k(x)\sqrt{8-x^2}dx,$$

then

$$\begin{aligned} \langle f, g\rangle &= \sum_{k=0}^{\infty} f_k g_k 2\gamma_{k+1} &(63) \\ &= \frac{1}{8\pi^3}\int_{-2\sqrt{2}}^{2\sqrt{2}}\int_{-2\sqrt{2}}^{2\sqrt{2}} f(x)g(y)\sqrt{8-x^2}\sqrt{8-y^2}\left[\pi\sum_{k=0}^{\infty}U_k(x)U_k(y)\gamma_{k+1}\right]dxdy \\ &= \frac{1}{8\pi^3}\int_{-2\sqrt{2}}^{2\sqrt{2}}\int_{-2\sqrt{2}}^{2\sqrt{2}} f(x)g(y)\sqrt{8-x^2}\sqrt{8-y^2}F(x,y)dxdy \end{aligned}$$

where

$$F(x,y) = \pi\sum_{k=0}^{\infty}U_k(x)U_k(y)\gamma_{k+1} = 2\int_{-\infty}^{\infty}\frac{z-z^3}{2(1-z^2)^2 + z^2(x^2+y^2) - z(1+z^2)xy}ds \quad (64)$$

with $z = \frac{\sin s}{s}$. Now (63) holds due to (62) and orthogonality of Chebyshev polynomial with respect to the Wigner semicircular law, that is,

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} U_n(x)U_m(x)\frac{1}{4\pi}\sqrt{8-x^2}dx = \delta_{mn}.$$

And (64) is a straightforward consequence of the Fourier analysis using the following fact

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}, \ x = 2\sqrt{2}\cos\theta.$$

This completes the proof of Proof of (29). □

Recent development

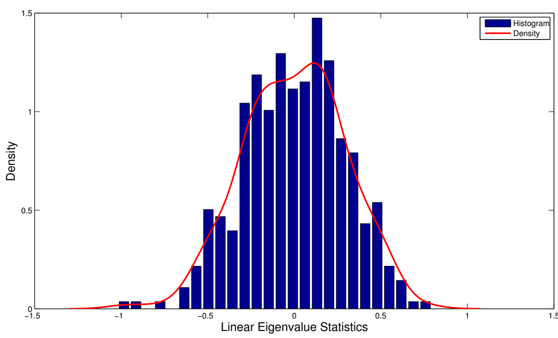
Recently, after submission of our paper, M. Shcherbina [?] improved our result by removing the restriction $b_n >> \sqrt{n}$ and proved it for all b_n which satisfies $b_n \to \infty$ and $\frac{b_n}{n} \to 0$ as $n \to \infty$.

Acknowledgment

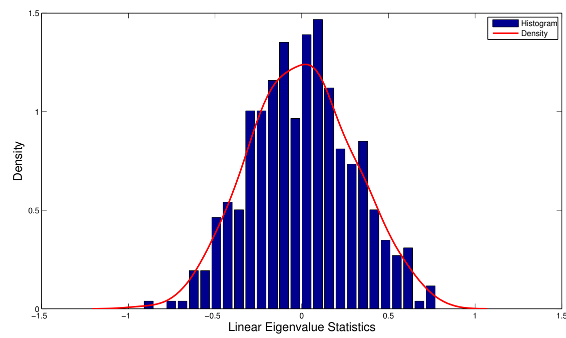
We thank the referee for her/his constructive comments which have led to a significant improvement in presentation.

5 Some MATLAB simulation results

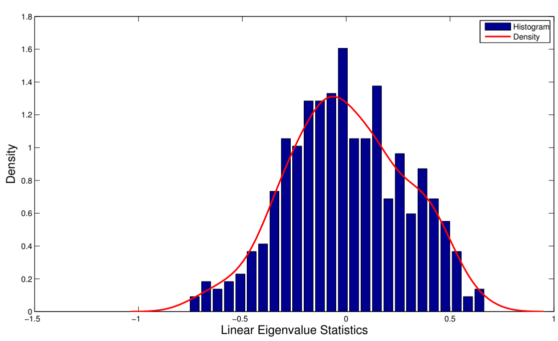
Here is what we found in MATLAB simulations.



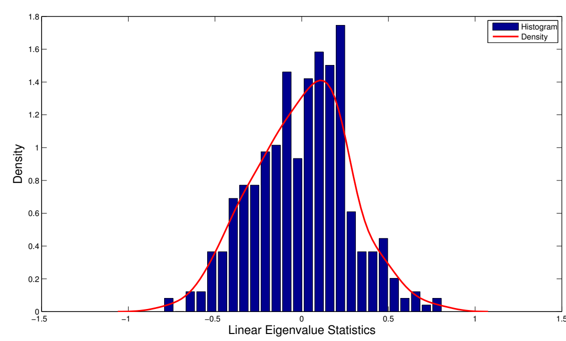
(a) $n = 2000$, $b_n = n^{0.2}$. Fourth moment/(variance)^2=2.92



(b) $n = 2000$, $b_n = n^{0.4}$. Fourth moment/(variance)^2=2.71



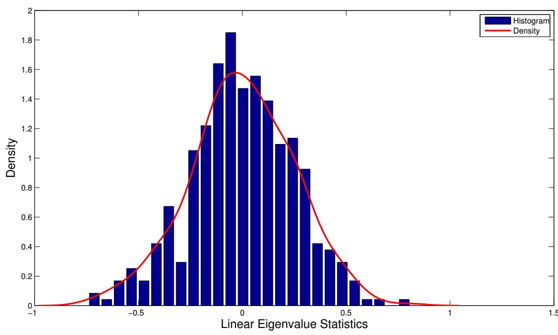
(a) $n = 2000$, $b_n = n^{0.6}$. Fourth moment/(variance)^2=2.57



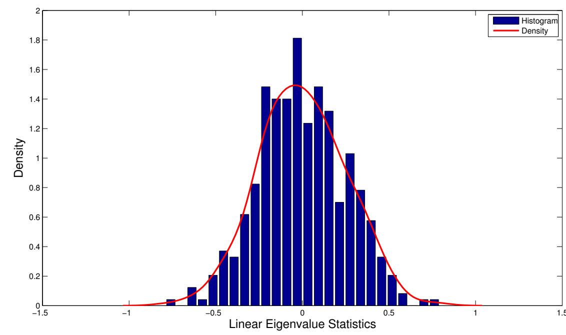
(b) $n = 2000$, $b_n = n^{0.8}$. Fourth moment/(variance)^2=2.91

Figure 4: The eigenvalue statistics was sampled 400 times. The test function was $\phi(x) = \sqrt{16 - x^2}$.

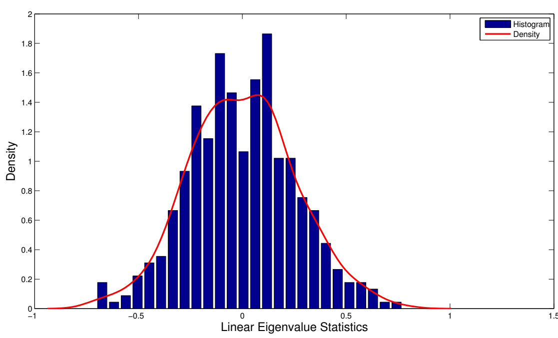
In the following example we had taken a different test function.



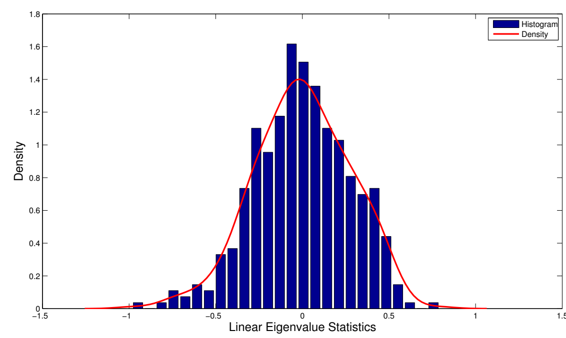
(a) $n = 2000$, $b_n = n^{0.2}$. Fourth moment/(variance)^2=3.08



(b) $n = 2000$, $b_n = n^{0.4}$. Fourth moment/(variance)^2=2.94



(c) $n = 2000$, $b_n = n^{0.6}$. Fourth moment/(variance)^2=3.00



(d) $n = 2000$, $b_n = n^{0.8}$. Fourth moment/(variance)^2=3.08

Figure 5: The eigenvalue statistics was sampled 400 times. The test function was $\phi(x) = e^{-x^2}$.