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Splitting of the Low Landau Levels into a Set of Positive Lebesgue Measure under Small Periodic Perturbations

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Dedicated to the memory of Roland Dobrushin

Abstract: We study the spectral properties of a two-dimensional Schrödinger operator with a uniform magnetic field and a small external periodic field:

$$L_{\varepsilon_0}(B) = -\frac{1}{2}\left[\left(\frac{\partial}{\partial x} - iBy\right)^2 + \frac{\partial^2}{\partial y^2}\right] + \varepsilon_0 V(x,y),$$

where

$$V(x,y) = V_0(y) + \varepsilon_1 V_1(x,y),$$

and ε_0, ε_1 are small parameters. Representing L_{ε_0} as the direct integral of one-dimensional quasi-periodic difference operators with long-range potential and employing recent results of E.I.Dinaburg about Anderson localization for such operators (we assume $2\pi/B$ to be typical irrational) we construct the full set of generalised eigenfunctions for the low Landau bands. We also show that the Lebesgue measure of the low bands is positive and proportional in the main order to ε_0.

1. Introduction

Spectral properties of Schrödinger operator describing electrons in the magnetic field have received a special attention recently in connection with attempts to explain Quantum Hall Effect ([1]-[8]). D.Thouless et al in [1] considered a two-dimensional model with constant magnetic field and a small periodic external field. In the Landau gauge it leads to the operator

$$L_{\varepsilon_0}(B) = -\frac{1}{2}\left[\left(\frac{\partial}{\partial x} - iBy\right)^2 + \frac{\partial^2}{\partial y^2}\right] + \varepsilon_0 V(x,y), \tag{1.1}$$

where B is the value of magnetic field, $V(x, y)$ is a smooth enough 1-periodic function, ε_0 is a small parameter. (In [1] the case of external potential $\alpha \cos(2\pi x) + \beta \cos(2\pi\tau y)$ was considered.)

If ε_0 equals zero, the spectrum $\sigma(L_0(B))$ of $L_0(B)$ consists of the discrete sequence of numbers:

$$\lambda_m = (m + \frac{1}{2})B, \quad m \in \mathbb{Z}^1_+, \tag{1.2}$$

($\mathbb{Z}^1_+$ is the set of nonnegative integers) called Landau levels ([10]). Each level is infinitely degenerate and the differential operator (1.1) leaves invariant the subspace of functions $\exp(2\pi ipx)\Psi(y),\ p \in \mathbb{R}^1$ since

$$L_0(B)(\exp(2\pi ipx)\Psi(y)) = \exp(2\pi ipx) \cdot \left\{ -\frac{1}{2}\frac{d^2}{dy^2} + \frac{B^2}{2}(y - 2\pi B^{-1}p)^2 \right\} \Psi(y). \tag{1.3}$$

In other words, if we consider $L^2(\mathbb{R}^2)$ as a direct integral of $L^2(\mathbb{R}^1)$:

$$L^2(\mathbb{R}^2) = \bigoplus \int_{-\infty}^{\infty} \tilde{H}_p dp, \quad \tilde{H}_p \sim L^2(\mathbb{R}^1),$$

where $\tilde{H}_p$ consists of the functions $\hat{f}(p, y)$ given by Fourier transform

$$f(x, y) = \int_{-\infty}^{\infty} e^{2\pi ipx} \hat{f}(p, y) dp,$$

then $L_0(B)$ is equal to the direct integral of shifted harmonic oscillators:

$$L_0 = \int_{-\infty}^{\infty} \tilde{L}_{0,p} dp,$$

$$\tilde{L}_{0,p} = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{B^2}{2}(y - 2\pi B^{-1}p)^2$$

(For the definition and properties of the direct integral see [11], vol.1 ch.2.1, vol. 4 ch. 13.16.). The eigenfunctions of $\tilde{L}_{0,p}$ are

$$\left\{ B^{\frac{1}{4}} \Omega_m \left(B^{\frac{1}{2}}(y - 2\pi B^{-1}p) \right) \right\} \ m \in \mathbb{Z}^1_+, \tag{1.4}$$

where Ω_m are Weber-Hermite functions: $\Omega_m(y) = \frac{(-1)^m}{\pi^{\frac{1}{4}}(2^m m!)^{\frac{1}{2}}} \exp\left(\frac{y^2}{2}\right) \frac{d^m}{dy^m} \exp(-y^2)$.

The eigensubspace, corresponding to the $m^{\underline{th}}$ Landau level is denoted by $E_0^{(m)}$. It follows from the general theory of perturbations that for $\varepsilon_0 << 1$ the operator $L_{\varepsilon_0}(B)$ has invariant subspaces $E_{\varepsilon_0}^{(m)}$, close to $E_0^{(m)}$. In this paper we will study the spectrum of the restriction of $L_{\varepsilon_0}(B)$ to $E_{\varepsilon_0}^{(m)}, \quad m < M(\varepsilon_0, B, V(x, y))$. If external potential $V(x, y)$ depends only on y, L_{ε_0} is still periodic in x and exibits localization in the y direction:

$$
\begin{aligned}
&L_{\varepsilon_0}(B)\,(\exp(2\pi ipx)\Psi(y))\\
= \quad &\exp(2\pi ipx)\cdot\left\{-\frac{1}{2}\frac{d^2}{dy^2}+\frac{B^2}{2}(y-2\pi B^{-1}p)^2+\varepsilon_0V(y)\right\}\Psi(y)
\end{aligned}
$$

Under such periodic perturbation each Landau level λ_m transforms into some interval of length $const_m\varepsilon_0$ located in a $O(\varepsilon_0)$ neighborhood of λ_m. The band function, which we denote by $\Lambda^{(m)}(p)$ is the m^{th} eigenvalue of a quantum Hamiltonian

$$
-\frac{1}{2}\frac{d^2}{dy^2}+\frac{B^2}{2}y^2+\varepsilon_0V(y+2\pi B^{-1}p)
$$

The corresponding eigenfunctions are decaying superexponentially in the y direction. The function $\Lambda^{(m)}$ will be as smooth as we want (or even analytic) if we assume the smoothness (analyticity) condition on V(see [11] vol.4 ch.12 , [24]). The aim of our paper is to extend the study of the low Landau bands to the case

$$
V(x,y)=V_0(y)+\varepsilon_1V_1(x,y) \tag{1.5}
$$

where ε_1 is a small parameter. We assume that V_0, V_1 are smooth enough:

$$
\textbf{(C)}\qquad \left\{\begin{array}{c}
V_0(y)\in C^6(S^1);\ \ V_1 \ \text{ has continuous derivatives}\\
\frac{\partial^iV_1}{\partial y^i},\ i\le 7\ \text{ in the cube}\\
\{x:\ |Im\,x|<\delta\}\times\{y:\ 0\le y\le 1\};\\
\frac{\partial^7V_1}{\partial y^7}(x,y)\ \text{ is analytic in the strip}\\
|Im\,x|<\delta\ \text{ for any fixed }\ y.
\end{array}\right.
$$

δ is some positive number, $\varepsilon_1 << 1$.
Some of our results are valid under stronger conditions on the smoothness of V_0, V_1:

$$
\textbf{(C}^*\textbf{)}\qquad \left\{\begin{array}{c}
V_0\in C^\infty(S^1),\ \ V_1\in C^\infty(T^2);\\
\text{all derivatives}\,\frac{\partial^iV_1}{\partial y^i}\ \text{ are analytic in the strip}\\
|Im\,x|<\delta.
\end{array}\right.
$$

The spectrum of (1.1) depends on the arithmetic nature of $\omega = 2\pi B^{-1}$. The case of rational ω was fully investigated by S.P. Novikov ([12]) and B.A. Dubrovin ,S.P. Novikov ([13],[14]). We study below the case of typical (Diophantine) irrational ω; i.e.

(D) $\quad |\{n\cdot\omega\}| > \frac{C}{|n|^\kappa};\qquad n\in\mathbb{Z}^1\backslash 0$

for some constants $C > 0, \kappa > 1$. Below we represent the restriction of $L_{\varepsilon_0}(B)$ to $E^{(m)}_{\varepsilon_0}$ as the direct integral of difference operators on the lattice with quasi-periodic coefficients which allow us to apply known results about Anderson localization for such operators (see [15–18]). We are able to construct the full family of generalized eigenfunctions $\{\Phi_q\}_{q\in R^1}$ for the low Landau levels. The corresponding band functions $\Lambda^{(m)}$ are ε_0^2-close to the band functions of the x-periodic operator obtained by setting $\varepsilon_1 = 0$. For $\varepsilon_1 \neq 0$ we prove polynomial localization in the y direction. We formulate our main results in Sect. 2 (Theorem 3). Proposition 1 and Theorem 2 are of more auxiliary nature.

2. Formulation of the Main Results

If $\varepsilon_1 \neq 0$ the differential operator (1.1) no longer leaves invariant the subspace of functions $\exp(2\pi ipx)\Psi(y)$. Nevertheless the image of any linear combination

$$\sum_n \exp(2\pi i(p+n)x)\Psi_n(y) \tag{2.1}$$

is again a function of this type. Choosing in the space of functions (2.1) the basis

$$\left\{\exp(2\pi i(p+n)x)B^{\frac{1}{4}}\Omega_m\left(B^{\frac{1}{2}}(y-(p+n)\omega)\right)\right\},$$

the double index (m,n) runs through $\mathbb{Z}^1_+ \times \mathbb{Z}^1$, we arrive at

Proposition 1. *The Hilbert space $L^2(\mathbb{R}^2)$ can be represented as a direct integral of $l^2\left(\mathbb{Z}^1_+ \times \mathbb{Z}^1\right)$*

$$L^2(\mathbb{R}^2) = \bigoplus\int_0^\omega H_p dp, \quad H_p \sim l^2\left(\mathbb{Z}^1_+ \times \mathbb{Z}^1\right)$$

such that the Schrödinger operator (1.1) equals the direct integral of difference operators $L_{\varepsilon_0,p}$ acting on $l^2\left(\mathbb{Z}^1_+ \times \mathbb{Z}^1\right)$:

$$L_{\varepsilon_0}(B) = \bigoplus\int_0^\omega L_{\varepsilon_0,p}dp,$$

where for $h(m,n) \in l^2\left(\mathbb{Z}^1_+ \times \mathbb{Z}^1\right)$,

$$\begin{aligned}\left(L_{\varepsilon_0,p}h\right)(m,n) = &\left[(m+\frac{1}{2})B + \varepsilon_0 V_{m,m}(p+n\omega)\right]h(m,n)+\\ &+\sum_{\substack{m_1\neq m,\\ m_1\geq 0}} \varepsilon_0 V_{m_1,m}(p+n\omega)h(m_1,n)+\\ &+\sum_{k=-\infty}^{\infty}\varepsilon_0\varepsilon_1\sum_{m_1=0}^{\infty} W^{(k)}_{m_1,m}(p+n\omega)h(m_1,n-k)\end{aligned} \tag{2.2}$$

In these expressions

$$V_{m_1,m}(\alpha) = B^{\frac{1}{2}}\int_{-\infty}^{\infty} V_0(y+\alpha)\Omega_{m_1}(B^{\frac{1}{2}}y)\Omega_m(B^{\frac{1}{2}}y)dy, \tag{2.3}$$

$$W^{(k)}_{m_1,m}(\alpha) = B^{\frac{1}{2}}\int_{-\infty}^{\infty} V^{(k)}_1(y+\alpha)\Omega_{m_1}(B^{\frac{1}{2}}(y+k\omega))\Omega_m(B^{\frac{1}{2}}y)dy, \tag{2.4}$$

$$V_1(x,y) = \sum_{k=-\infty}^{\infty}\exp(2\pi ikx)V^{(k)}_1(y). \tag{2.5}$$

The subspace, corresponding to the $m^{\underline{th}}$ Landau level is generated by the vectors $\{\delta_{m,m_1}\delta_{n,n_1}\}_{n\in\mathbb{Z}^1}$. If we denote the projection to it by $P^{(m)}_{0,p}$ then , as was first observed by D. Hofstadter [9]

$$\begin{aligned}\left(P^{(m)}_{0,p}L_{\varepsilon_0,p}P^{(m)}_{0,p}h\right)(m,n) &= \left[(m+\frac{1}{2})B+\varepsilon_0 V_{m,m}(p+n\omega)\right]h(m,n)+\\ &+\varepsilon_0\varepsilon_1\sum_k W^{(k)}_{m,m}(p+n\omega)h(m,n-k)\end{aligned} \tag{2.6}$$

is the one-dimensional difference operator with exponentially decaying quasiperiodic coefficients. (If $V(x,y)=\alpha\cos(2\pi y)+\beta\cos(2\pi\tau x)$ (2.6) is just the Almost Mathieu operator.) It turns out that one can find such a unitary operator $U(p)$ that the restrictions of $U(p)^{-1}L_{\varepsilon_0,p}U(p)$ to the invariant subspaces $E^{(m)}_{\varepsilon_0,p},\quad m\le M\,(B,\varepsilon_0,V(x,y))$ have the form similar to the r.h.s. of (2.6). This is the main result of Theorem 2.

Theorem 2. *Assume that the parameters $\varepsilon_0,\varepsilon_1$ are small enough. Then there exists an integer $M=M\,(B,\varepsilon_0,V)$ tending to ∞ as $\varepsilon_0,\varepsilon_1\to 0$ so that for any $m\le M$ the restriction of $L_{\varepsilon_0}(B)$ to $E^{(m)}_{\varepsilon_0}$ is the direct integral of one-dimensional difference operators with exponentially decaying coefficients:*

$$L_{\varepsilon_0}(B)|_{E^{(m)}_{\varepsilon_0}}=\bigoplus\int_0^\omega L^{(m)}_{\varepsilon_0}(p)dp,$$

where for $\varphi\in l^2\left(\mathbb{Z}^1\right)$,

$$\left(L^{(m)}_{\varepsilon_0}(p)\varphi\right)(n)=d_m(p+n\omega)\varphi(n)+\sum_{k\neq n}a_m(n-k,p+n\omega)\varphi(k) \tag{2.7}$$

$$\left\|d_m-(m+\frac{1}{2})B-\varepsilon_0 V_{m,m}\right\|_{C^2(S^1)}<const_1\varepsilon_0^2 \tag{2.8}$$

$$\sum_{k\neq 0}\|a_m(k,p)\|_{C^2(S^1)}e^{\frac{2\delta}{3}|k|}<const_2\varepsilon_1\varepsilon_0 \tag{2.9}$$

The proof of Theorem 2 uses standard methods of perturbation theory. However, some nontrivial details due to the special form of the operator $L_{\varepsilon_0,p}$ remain. The proof is given in Sects. 3, 4.

The family $L^{(m)}_{\varepsilon_0}(p)$ is an ergodic family of operators in the sense of [18], associated with the dynamical system $\left\{S^1,T_\omega,l\right\}$ and defined by some function $h^{(m)}(n,p)$. Here S^1 is the unit circle, T_ω is the rotation $x\to(x+\omega)mod1, l$ is the Lebesgue measure and $h^{(m)}(n,p)$ is a function of two variables $n\in\mathbb{Z}^1,\ p\in S^1$, such that

$$h^{(m)}(n,p)=d_m(p),\quad\text{for}\quad n=0,\quad h^{(m)}(n,p)=a_m(n,p),\quad\text{for}\quad n\neq 0.$$

The matrix elements of $L^{(m)}_{\varepsilon_0}(p)$ in the natural basis are given by the formula

$$L^{(m)}_{\varepsilon_0}(p)_{kl}=h^{(m)}(l-k,T^k_\omega p).$$

It follows from Theorem 2 that if $d^{(m)}(p)$ is a Morse function on the unit circle, having two critical points, the family $L^{(m)}_{\varepsilon_0}(p)$ satisfies the conditions of the main theorem from

[18]. For completeness we formulate this theorem below. Let $h(n,p) \in C^2(S^1)$ for any $n \in \mathbb{Z}^1$; $h(0,p)$ be a Morse function with two critical points,

$$\sum_{n\neq 0} \|h(n,p)\|_{C^2(S^1)}\, e^{\rho|n|} < \varepsilon \ \text{ for some } \ \rho > 0,$$

and ω satisfy the Diophantine condition (**D**) with constants C, κ.

Theorem (18). *One can find $\bar{\varepsilon} = \bar{\varepsilon}(C,\kappa,h(0,p),\rho)$ so that for any $|\varepsilon| < \bar{\varepsilon}$,*

a) *For a.e. p (with respect to Lebesgue measure), the spectrum of $L^{(m)}_{\varepsilon_0}(p)$ is pure point, its eigenvalues coincide with the values of some function $\Lambda(p) \in L^\infty(S^1)$ along the trajectory $\{T^n_\omega p\}_{n\in\mathbb{Z}^1}$ of the point p.*
b) *The corresponding eigenfunctions decay exponentially. They can be constructed with the help of a function $\Psi(n,p)$, measurable for any $n \in \mathbb{Z}^1$, such that for a.e. p $\sum\limits_n |\Psi(n,p)|\, e^{\frac{\rho}{2}|n|} < \infty$. Then the set of eigenfunctions $\{\varphi_k\}_{k=-\infty}^{+\infty}$ is given by the formula $\varphi_k(n) = \Psi(n-k, p+k\omega)$.*
c) *The spectrum is nondegenerate.*
d) *The spectrum as a set (i.e. the closure of the set of eigenvalues) has a positive Lebesgue measure, greater than $l(Ran(h(0,p)) - const\cdot\varepsilon^\sigma, \sigma > 0$.*

Combining this result and the statement of Theorem 2 we arrive at

Theorem 3. *Let $V_{m,m}(p) = B^{\frac{1}{2}} \int\limits_{-\infty}^{\infty} V_0(y+p)\Omega^2_m(B^{\frac{1}{2}}y)dy$, $m \le M(B,\varepsilon_0,V)$ be a Morse function with two critical points and $\omega = 2\pi B^{-1}$ satisfies (* **D** *). Then there exist positive constants $\bar{\varepsilon}_0, \bar{\varepsilon}_1(B,V_0,V_1,C,\kappa,m)$ such that for $|\varepsilon_0| < \bar{\varepsilon}_0$, $|\varepsilon_1| < \bar{\varepsilon}_1$ the following statements hold:*

i) For any fixed $s \in \mathbb{Z}^1_+$, $n \in \mathbb{Z}^1$ there exist functions $c^{(m)}(s,n;p)$, 1-periodic and measurable in p and 1-periodic measurable functions $\Lambda^{(m)}(p)$ such that

$$\left\| \Lambda^{(m)}(p) - ((m+\frac{1}{2})B + \varepsilon_0 V_{m,m}(p)) \right\|_{L^\infty(S^1)} < const\cdot\varepsilon_0^2;$$

for a.e. $p \in [0,1]$

$$\sum_{s,n} \left|c^{(m)}(s,n;p)\right| (s^2+1) e^{\frac{\delta}{3}|n|} < \infty; \tag{2.10}$$

and for every $k \in \mathbb{Z}^1$, a.e. $p \in [0,\omega]$ the series

$$\Phi^{(m)}_{p,k}(x,y) = \sum_{s,n} c^{(m)}(s,n-k,p+k\omega) e^{2\pi i(\frac{p}{\omega}+n)x}\cdot \Omega_s(B^{\frac{1}{2}}(y-p-n\omega)) \tag{2.11}$$

and their first and second derivatives converge uniformly in x, y giving generalized eigenfunctions of $L_{\varepsilon_0}(B)$ with the eigenvalues $\Lambda^{(m)}(p+k\omega)$,

$$L_{\varepsilon_0}(B)\Phi^{(m)}_{p,k}(x,y) = \Lambda^{(m)}(p+k\omega)\Phi^{(m)}_{p,k}(x,y)$$

The constructed functions $\Phi^{(m)}_{p,k}(x,y)$ are infinitely differentiable in x;

$\Phi^{(m)}_{p,k}(x+1,y) = e^{ipB}\Phi^{(m)}_{p,k}(x,y)$, and they decay at infinity in y at least as $\frac{1}{y^2+1}$

$$\left|\Phi^{(m)}_{p,k}(x,y)\right| \leq \frac{const(p,k,m)}{y^2+1}. \tag{2.12}$$

If the functions $V_0,\ V_1$ satisfy the stronger condition ($\mathbf{C}^*$), then for any integer $N > 0$ one can find so small $\bar{\varepsilon}_0, \bar{\varepsilon}_1$ (depending on N) that for a.e. p the functions $\Phi_{p,k}(x,y)$ are infinitely differentiable in x and y and

$$\left|\Phi^{(m)}_{p,k}(x,y)\right| \leq \frac{const(p,k,m)}{y^{2N}+1}. \tag{2.13}$$

(ii) One can construct a full family of eigenfunctions defining them for any real parameter $q \in \mathbb{R}^1$ by the formula

$$\Phi^{(m)}_q(x,y) = \Phi^{(m)}_{\{\frac{q}{\omega}\}\omega,[\frac{q}{\omega}]}(x,y)$$

so that for any $f(x,y)$ from the Schwartz space $J(\mathbb{R}^2)$ the Plancherel formula holds:

$$\left\|P^{(m)}_{\varepsilon_0} f\right\|^2 = \int_{-\infty}^{\infty} |g_f(q)|^2 dq,$$

where $P^{(m)}_{\varepsilon_0}$ is the projection to $E^{(m)}_{\varepsilon_0}$ and

$$g_f(q) = \int_{\mathbb{R}^2} f(x,y)\overline{\Phi_q(x,y)}dxdy.$$

(iii) The restriction of $L_{\varepsilon_0}(B)$ to $E^{(m)}_{\varepsilon_0}$ is unitary equivalent to the multiplication operator on $L^2(\mathbb{R}^1)$ with the multiplication function $\Lambda^{(m)}$; the Lebesgue measure of the m^{th} Landau band equals to $\varepsilon_0 l(Ran(V_{m,m})) + o(\varepsilon_0)$.

Remark. The nature of the spectrum clearly depends on the type of the distribution function of $\Lambda^{(m)}$. For the Almost Mathieu operator the distribution function of $\Lambda^{(m)}$ is known to be absolute continuous (see [15]). However for the general quasi-periodic operators with long-range potential, studied in [18], this is still an open question.

Remark. The condition on $V_{m,m}(p)$ formulated in the Theorem 3 is satisfied for example by $V_0(y) = \cos(2\pi y)$.

Remark. B.Helffer and J.Sjöstrand applied in [19, 20] the semiclassical analysis of the Almost Mathieu equation to the case of the Schrödinger operator with a strong symmetric external field ($\varepsilon_0 >> 1$). They showed (under some conditions on the continuous fraction expansion of $2\pi/B$) that in the neighborhood of the first eigenvalue of the approximating hamiltonian with a quadratic potential, the spectrum of L is a Cantor set of zero Lebesgue measure.

We will discuss Theorem 3 in more detail in Sect. 5.

3. Reduction of the Matrix Representation of $L_{\varepsilon_0,p}$ in the Neighborhood of the Low Landau Levels to the Special Block Type

We will prove Theorem 2 in the case of the lowest Landau level. The generalization to the case of a few Landau levels is straightforward.

Let us write the matrix of $L_{\varepsilon_0,p}$ in the block form:

$$\begin{pmatrix} (0,0) & (0,1) & (0,2) & (0,3) & \cdots \\ (1,0) & (1,1) & (1,2) & \cdots & \cdots \\ (2,0) & (2,1) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.1}$$

which consists of the countable number of blocks, enumerated by the double index (m, m_1), $\quad m \in \mathbb{Z}^1_+, \quad m_1 \in \mathbb{Z}^1_+$. Each block is infinitely-dimensional and its matrix elements correspond to the interaction between the $m^{\underline{th}}$ and $m_1^{\underline{th}}$ Landau levels. In this special representation we are looking for a unitary operator $U(p)$, such that the matrix $U(p)^{-1}L_{\varepsilon_0,p}U(p)$ has zero non-diagonal blocks $(0, m_1), (m_1, 0)$ for $m_1 \neq 0$, and block $(0,0)$ is given by an operator of the type (2.7–2.9). We represent the matrix of $L_{\varepsilon_0,p}$ as $L = D_{(1)} + A^{(1)}_{(1)} + A^{(2)}_{(1)}$, where $D_{(1)}$ is a diagonal part,

$$D_{(1)}(m, n; m_1, n_1) = \delta_{m,m_1}\delta_{n,n_1}L(m, n; m_1, n_1),$$

$A^{(2)}_{(1)}$ corresponds to the interaction of the zero Landau level and the other levels,

$$A^{(2)}_{(1)}(m, n; m_1, n_1) = \delta_{0,m}(1-\delta_{0,m_1})L(m, n; m_1, n_1) + (1-\delta_{0,m})\delta_{0,m_1}L(m, n; m_1, n_1),$$

and $A^{(1)}_{(1)} = L - D - A^{(2)}_{(1)}$. We can write the conditions on $U = e^{iW}$ as

$$\left.\begin{array}{c} \left(e^{-iW}Le^{iW}\right)(0, n; m_1, n_1; p) = 0 \ \text{ if } \ m_1 > 0 \\ \sum\limits_{n_1} \left\| \left(e^{-iW}Le^{iW}\right)(0, n; 0, n_1; p) \right\|_{C^2(S^1)} e^{\frac{2}{3}\delta|n_1-n|} < +\infty \end{array}\right\}. \tag{3.2}$$

We also require $W(p)$ to be an ergodic family of operators: $W(m, n; m_1, n_1; p) = W(m, 0; m_1, n_1 - n; p + n\omega)$. To define W we use the well known formula

$$e^{-iW}Le^{iW} = L + \sum_{k=1}^{\infty} \frac{i^k}{k!} \underbrace{[\cdots[L, W], \cdots W]}_{k \ \text{times}}.$$

In the first approximation $W_{(1)}$ is the solution of the equation

$$i[D_{(1)}, W_{(1)}] = -A^{(2)}_{(1)}, \tag{3.2}$$

i.e. for $m_1 \leq m_2$

$$\left.\begin{array}{l} W_{(1)}(m_1, n_1; m_2, n_2) = 0 \ \text{ if } \ m_1 > 0 \\ W_{(1)}(0, n_1; m_2, n_2) = i\dfrac{A^{(2)}_{(1)}(0, n_1; m_2, n_2)}{D(0, n_1; 0, n_1) - D(m_2, n_2; m_2, n_2)} \end{array}\right\}. \tag{3.4}$$

Then for $L_{(2)} = e^{-iW_{(1)}}Le^{iW_{(1)}}$ we obtain the analogous representation $L_{(2)} = D_{(2)} + A^{(1)}_{(2)} + A^{(2)}_{(2)}$ in which $A^{(2)}_{(2)}$ has a norm of order ε_0^2. In the same way we can find the next approximation solving the equation $i\left[D_{(2)}, W_{(2)}\right] = -A^{(2)}_{(2)}$, and so on. It is clear that

$$|D_{(1)}(0,n_1;0,n_1) - D_{(1)}(m_2,n_2;m_2,n_2)| >$$
$$> m_2\frac{B}{2} - \varepsilon_0\left(\|V_{0,0}\|_{C^2(S^1)} + \|V_{m_2,m_2}\|_{C^2(S^1)}\right) > m_2\frac{B}{4}$$

if $m_2 > 0$ and ε_0 is small enough.

The same inequality for $D_{(s)}$ immediately follows from the inductive assumptions $(I_s - III_s)$ (see below). It means that the small denominators do not appear on each step of our inductive procedure and the standard perturbation theory can be applied. The most convenient way to formulate the inductive hypothesises is to use the functions

$$l(m_1,m_2;n,p) := L(m_1,0;m_2,n;p), \text{ such that}$$
$$L(m_1,n_1;m_2,n_2;p) = l(m_1,m_2;n_2-n_1;p+n_1\omega).$$

Remark also that the product of two ergodic operators B, C corresponds to the convolution of functions b, c:

$$(b\cdot c)(m_1,m_2;n_1,p) = \sum_{m,n} b(m_1,m;n,p)\cdot c(m,m_2;n_1-n;p+n\omega).$$

Now we are ready to formulate the inductive assumptions at the $s^{\underline{th}}$ step of induction:

$\underline{(I_{s,l})}$

$$a)\quad \sum_{m_1=0}^{\infty}\left\|a^{(1)}_{(s)}(m,m_1;0;p)\right\|_{C^2(S^1)}\cdot\left(m_1^l+1\right) \le \left(m^{l+1}+1\right)\delta_{(s)},$$

$$b)\quad \sum_{m_1=0}^{\infty}\left(\sum_{n\neq 0}\left\|a^{(1)}_{(s)}(m,m_1;n;p)\right\|_{C^2(S^1)}\cdot e^{\frac{2}{3}\delta|n|}\right)\cdot\left(m_1^l+1\right) \le \varepsilon_1\left(m^{l+1}+1\right)\delta_{(s)}.$$

$\underline{(II_{s,l})}$

$$a)\quad \sum_{m_1}\left\|a^{(2)}_{(s)}(0,m_1;0;p)\right\|_{C^2(S^1)}\cdot\left(m_1^l+1\right) < \varepsilon_{(s)},$$

$$b)\quad \sum_{m_1}\left(\sum_{n\neq 0}\left\|a^{(2)}_{(s)}(0,m_1;n;p)\right\|_{C^2(S^1)}\cdot e^{\frac{2}{3}\delta|n|}\right)\cdot\left(m_1^l+1\right) \le \varepsilon_1\varepsilon_{(s)}.$$

$\underline{(III_{s,l})}$

$$\left\|d_{(s)}(m,m;0;p) - (m+\frac{1}{2})B - \varepsilon_0 V_{m,m}(p)\right\|_{C^2(S^1)} \le \varepsilon_0\delta_{(s)}.$$

We will see later that there exist some constants $const_{3,l}(V_0,V_1,B)$ and $const_{4,l}(V_0,V_1,B)$, such that

$$0 < \delta_{(s)} < const_{3,l}\cdot\varepsilon_0,$$
$$0 < \varepsilon_{(s)} < \left(const_{4,l}\cdot\varepsilon_0\right)^s.$$

Proposition 4. *The inductive assumptions* $\left(I_{1,1} - III_{1,1}\right)$ *are valid for* $s = 1$ *and* $\varepsilon_{(1)} = \delta_{(1)} = const_{5,1}(V_0, V_1, B) \cdot \varepsilon_0$. *If* V_0, V_1 *satisfy* $(\mathbf{C}^*)$ *, inductive assumptions* $\left(I_{1,l} - III_{1,l}\right)$ $l = 1, 2\ldots$ *are valid at the first step of induction* $s = 1$ *with* $\varepsilon_{(1)} = \delta_{(1)} == const_{5,l}(V_0, V_1, B) \cdot \varepsilon_0$.

Remark. Various constants appearing in the proof of Theorems 2,3 depend only on the magnetic field and external potential $V(x, y)$. Proposition 4 will be proven in Sect. 4.

Lemma 5. *Assume that the inductive assumptions* $\left(I_{s,l} - III_{s,l}\right)$ *are valid on the* $s^{\underline{th}}$ *step of induction. Define* $W_{(s)}$ *with the help of the formula*

$$\left[D_{(s)}, W_{(s)}\right] = iA^{(2)}_{(s)}.$$

Then the inductive assumptions $\left(I_{s+1,l} - III_{s+1,l}\right)$ *are valid for*

$$L_{(s+1)} = e^{-iW_{(s)}} L_{(s)} e^{iW_{(s)}},$$
$$\varepsilon_{(s+1)} = \varepsilon_{(s)} \cdot const_{7,l} \cdot \delta_{(s)}, \quad \delta_{(s+1)} = \delta_{(s)} \left(1 + const_{6,l}\varepsilon_{(s)}\right).$$

Moreover

$$w_{(s)}(0, m; n) \sim \frac{a^{(2)}_{(s)}(0, m; n)}{m + 1}.$$

Remark. The last relation allows us to write an additional power of m in the r.h.s. of inequalities $(I_{s,l})$.

The proof of inductive lemma is rather standard and will be omitted.

The operator $U(p) = e^{iW} = \lim_{s\to\infty} \prod e^{iW_{(s)}}$ is well defined and satisfies the statement of Theorem 2.

4. Checking the Inductive Assumptions at the First Step of Induction

Writing $L_{\varepsilon_0,p} = D + A^{(1)} + A^{(2)}$ we have the following representation for the matrix elements:

$$d(m, m; 0; p) = \left(m + \frac{1}{2}\right) B + \varepsilon_0 V_{m,m}(p) = \left(m + \frac{1}{2}\right) B + \\ + \varepsilon_0 B^{\frac{1}{2}} \int_{-\infty}^{\infty} V_0(y + p)\Omega_m \left(B^{\frac{1}{2}} y\right) \Omega_m \left(B^{\frac{1}{2}} y\right) dy, \tag{4.1}$$

$$a^{(1)}(m, m_1; n, p) = \varepsilon_0\varepsilon_1 W^{(n)}_{m,m_1}(p) = \\ = \varepsilon_0\varepsilon_1 B^{\frac{1}{2}} \int_{-\infty}^{\infty} V^{(n)}_1(y + p)\Omega_m \left(B^{\frac{1}{2}}(y + n\omega)\right) \Omega_{m_1} \left(B^{\frac{1}{2}} y\right) dy \tag{4.2}$$

if $n \neq 0$, where $V^{(n)}_1(y)$ are Fourier coefficients of $V_1(\cdot, y)$:

$$V_1(x, y) = \sum_{n=-\infty}^{\infty} e^{2\pi inx} V^{(n)}_1(y)$$

(clearly we can assume $V_1^{(0)}(y) \equiv 0$ or add it to $V_0(y)$),

$$a^{(1)}(m, m_1; 0, p) = \varepsilon_0 V_{m,m_1}(p) = \varepsilon_0 B^{\frac{1}{2}} \int_{-\infty}^{\infty} V_0(y+p)\Omega_m\left(B^{\frac{1}{2}}y\right)\Omega_{m_1}\left(B^{\frac{1}{2}}y\right)dy \tag{4.3}$$

if $m_1 \neq m \neq 0$;

$$a^{(2)}(0, m_1; 0, p) = \varepsilon_0 V_{0,m}(p) = \varepsilon_0 B^{\frac{1}{2}} \int_{-\infty}^{\infty} V_0(y+p)\Omega_0\left(B^{\frac{1}{2}}y\right)\Omega_{m_1}\left(B^{\frac{1}{2}}y\right)dy \tag{4.4}$$

if $m_1 > 0$; and

$$\begin{aligned} a^{(2)}(0, m_1; n, p) &= \varepsilon_0\varepsilon_1 W_{0,m_1}^{(n)}(p) = \\ &= \varepsilon_0\varepsilon_1 B^{\frac{1}{2}} \int_{-\infty}^{\infty} V_1^{(n)}(y+p)\Omega_0\left(B^{\frac{1}{2}}(y+n\omega)\right)\Omega_{m_1}\left(B^{\frac{1}{2}}y\right)dy \end{aligned} \tag{4.5}$$

if $n \neq 0,\ m_1 > 0$.

Conditions $\left(I_{1,l} - III_{1,l}\right)$ rewritten in terms of $V_{m,m_1}(p),\ W_{m,m_1}^{(n)}(p)$, lead to the inequalities:

$$\sum_{m_1} \|V_{m,m_1}\|_{C^2(S^1)}\left(m_1^l + 1\right) \leq const_{6,l}\left(m^{l+1} + 1\right), \tag{4.6}$$

$$\sum_{m_1}\left(\sum_n \left\|W_{m,m_1}^{(n)}\right\|_{C^2(S^1)} e^{\frac{2}{3}\delta|n|}\right)\left(m_1^l + 1\right) \leq const_{7,l}\left(m^{l+1} + 1\right). \tag{4.7}$$

The main part of the proof of estimates (4.6), (4.7) is contained in lemmas 6–7.

Lemma 6. *Let $\bar{m} \geq m,\ b \geq 0$. Then*

$$\begin{aligned} I(m, \bar{m}; b) &= \int_{-\infty}^{\infty} e^{iby}\Omega_m(y)\Omega_{\bar{m}}(y)dy = \\ &= i^{(\bar{m}-m)}\left(\frac{b}{\sqrt{2}}\right)^{(\bar{m}-m)} \cdot \frac{1}{((\bar{m}-m)!)^{\frac{1}{2}}} \cdot e^{-\frac{b^2}{4}} \cdot \begin{pmatrix} \bar{m} \\ m \end{pmatrix}^{\frac{1}{2}} \\ &\left\{\sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} (-1)^l \left(\frac{b^2}{2}\right)^l \cdot \frac{(\bar{m}-m)!}{(\bar{m}-m+l)!}\right\} \end{aligned} \tag{4.8}$$

Lemma 7.

$$a)\ \sum_{m_1=0}^{\infty} |I(m, m_1; b)| \cdot (m_1^l + 1) \leq 4\left(\max(4m; m + 18b^2)\right)^{l+\frac{1}{2}} + const_{8,l} \leq$$

$$\leq (12l + 12)\left((5m)^{l+\frac{1}{2}} + (3\sqrt{2}b)^{2l+1}\right) + const_{8,l}.$$

(b) Let the function $f(y)$ be periodic with period τ and $(2l+2)$ - times continuously differentiable. Then

$$\sum_{m_1=0}^{\infty} \sup_{\alpha} \left| \int_{-\infty}^{\infty} f(y+\alpha)\Omega_m(y)\Omega_{m_1}(y)dy \right| \left(m_1^l + 1\right) \leq$$
$$\leq \left\| f^{(2l+2)} \right\|_{L^2(S^1)} \cdot const_{9,l}(\tau) \cdot \left(m^{l+\frac{1}{2}} + 1\right).$$

Proof of Lemma 6.

$$I(0,\bar{m};b) = \int_{-\infty}^{\infty} e^{iby}\Omega_0(y)\Omega_{\bar{m}}(y)dy$$

is a well known integral (see [21, 22]):

$$I(0,\bar{m};b) = i^{\bar{m}} \left(\frac{b}{\sqrt{2}}\right)^{\bar{m}} \cdot \frac{e^{-\frac{b^2}{4}}}{(\bar{m}!)^{\frac{1}{2}}}. \tag{4.9}$$

It is not difficult to see that

$$I(m,\bar{m};b) = \sqrt{\frac{\bar{m}}{m}} I(m-1,\bar{m}-1;b) + i\frac{b}{\sqrt{2}}\sqrt{\frac{1}{m}} \cdot I(m-1,\bar{m};b). \tag{4.10}$$

Iterating (4.10) m times we arrive at (4.8).

Proof of Lemma 7.

$$a) \sum_{m_1=0}^{\infty} |I(m,m_1;b)| \cdot (m_1^l + 1) = \sum_{m_1=0}^{\max(4m,m+18b^2)} + \sum_{m_1 > \max(4m,m+18b^2)}.$$

We use a rough estimate for the first sum. Since

$$\sum_{m_1=0}^{\max(4m,m+18b^2)} |I(m,m_1;b)|^2 \leq \sum_{m_1=0}^{\infty} |I(m,m_1;b)|^2 = 1,$$
$$\sum_{m_1=0}^{\max(4m,m+18b^2)} |I(m,m_1;b)| \cdot \left(m_1^l + 1\right) \leq \left(\max(4m, m+18b^2) + 1\right)^{\frac{1}{2}} \cdot \tag{4.11}$$
$$\cdot \left((\max(4m, m+18b^2))^l + 1\right).$$

The second sum is uniformly bounded by a constant. To see this, we need

Lemma 8. *Let*

$$\bar{m} > \max(4m, m+18b^2). \tag{4.12}$$

Then

$$\left| \sum_{l=0}^{m} \binom{m}{l} (-1)^l \left(\frac{b^2}{2}\right)^l \frac{(\bar{m}-m)!}{(\bar{m}-m+l)!} \right| \leq 1.$$

Denoting $\left(\frac{b^2}{2}\right)^l \frac{(\bar{m}-m)!}{(\bar{m}-m+l)!}$ by $r(l)$ we can write

$$\sum_{l=0}^{m}(-1)^l \binom{m}{l} r(l) = \left(\bar{\Delta}^m r\right)(m),$$

where $\bar{\Delta} r(l) = r(l-1) - r(l), \quad l = 1, \cdots m$ and $\left(\bar{\Delta}^k r\right)(l) = \bar{\Delta}\left(\bar{\Delta}^{k-1} r\right)(l)$.

In our case

$$\bar{\Delta} r(l) = \left(\frac{2(\bar{m}-m)}{b^2} + \frac{2l}{b^2} - 1\right) r(l),$$

and, in general, for $t \le l$

$$\left(\bar{\Delta}^t r\right)(l) = \left(\frac{2(\bar{m}-m)}{b^2} + \frac{2(l+1-t)}{b^2} - 1\right)\bar{\Delta}^{t-1} r(l) - \frac{2}{b^2}\left(\bar{\Delta}^{t-2} r\right)(l). \quad (4.13)$$

Equations (3.12) and (3.13) imply $\left(\bar{\Delta}^t r\right)(l) > 0, \quad \left(\bar{\Delta}^t r\right)(l) > \left(\bar{\Delta}^{(t-1)} r\right)(l)$.

Finally $\left|\left(\bar{\Delta}^t r\right)(l)\right| \le r(l)\prod_{j=0}^{t-1}\left(\frac{2(\bar{m}-m)+2(l-j)}{b^2} - 1\right)$

and $\left|\left(\bar{\Delta}^m r\right)(m)\right| \le r(m)\prod_{j=0}^{m-1}\left(\frac{2(\bar{m}-j)}{b^2} - 1\right) \le \prod_{j=0}^{m-1}\left(1 - \frac{b^2}{2(\bar{m}-j)}\right) \le 1$. To finish the proof of Lemma 2 a) we write

$$\sum_{\bar{m} > \max(4m, m+18b^2)} |I(m,\bar{m},b)|\,(\bar{m}^l + 1) \le$$
$$\le \sum_{k > \max(3m, 18b^2)} \left(\frac{b}{\sqrt{2}}\right)^k \frac{1}{\sqrt{k!}} e^{-\frac{b^2}{4}} \binom{k+m}{m}^{\frac{1}{2}} \cdot$$
$$\left((k+m)^l + 1\right) \le \sum_{k > 18b^2} \left(\frac{b}{\sqrt{2}}\right)^k \frac{1}{\sqrt{k!}} e^{-\frac{b^2}{4}} 2^k \left((2k)^l + 1\right).$$

It is clear that the sum of the last series is uniformly bounded in b.

Part b) of Lemma 7 follows from part a) and estimates on decay of Fourier coefficients of differentiable functions. Lemma 2 is proven.

Remark. Since for Weber-Hermite functions

$$\left|\int_{-\infty}^{\infty} \Omega_m(y - n\omega)\Omega_{m_1}(y)dy\right| = \left|\int_{-\infty}^{\infty} e^{in\omega y}\Omega_m(y)\Omega_{m_1}(y)dy\right|,$$

the estimates from part a) of Lemma 7 imply

$$\sum_{n=-\infty}^{\infty} e^{-\frac{\delta}{12}|n|} \sum_{m_1=0}^{\infty} \left|\int_{-\infty}^{\infty} \Omega_m(y - n\omega)\Omega_{m_1}(y)dy\right| \cdot (m_1^l + 1) \le$$
$$\le const_{10,l} \cdot \left(m^{l+\frac{1}{2}} + 1\right). \quad (4.14)$$

Now we are ready to prove inequalities (4.6)–(4.7). The first of them immediately follows from Lemma 7 b). To check (4.7) we consider the Fourier series for $V_1(x,y)$:

$$V_1(x,y) = \sum_{n=-\infty}^{\infty} e^{2\pi inx} V_1^{(n)}(y) = \sum_{n,l} g(n,l) \cdot e^{2\pi inx} \cdot e^{2\pi iny}.$$

The condition (**C**) implies

$$|g(n,l)| \le e^{-\frac{3}{4}\delta|n|} \cdot \frac{1}{l^7+1} \cdot const(V_1).$$

Then

$$\begin{aligned}
&\sum_{m_1=0}^{\infty} \left(\sum_{n=-\infty}^{\infty} e^{\frac{2}{3}\delta|n|} \sup_{\alpha} \left| \int_{-\infty}^{\infty} \Omega_m \left(B^{\frac{1}{2}}(y-n\omega)\right) \cdot \right.\right. \\
&\left.\left. \cdot V_1^{(n)}(y+\alpha)\Omega_{m_1}\left(B^{\frac{1}{2}}y\right) dy \right| \right) \cdot (m_1+1) = \\
&= \sum_{m_1=0}^{\infty} \left(\sum_{n=-\infty}^{\infty} e^{\frac{2}{3}\delta|n|} \sup_{\alpha} \left| \int_{-\infty}^{\infty} \Omega_m \left(B^{\frac{1}{2}}(y-n\omega)\right) \cdot \right.\right. \\
&\left.\left. \left(\sum_{l} g(n,l) e^{2\pi ily} e^{2\pi il\alpha} \right) \Omega_{m_1}\left(B^{\frac{1}{2}}y\right) dy \right| \right) \cdot (m_1+1) = \\
&= \sum_{m_1=0}^{\infty} \left(\sum_{n=-\infty}^{\infty} e^{\frac{2}{3}\delta|n|} \sup_{\alpha} \left| \int_{-\infty}^{\infty} \sum_{\bar{m}=0}^{\infty} I(m,\bar{m},lB^{-\frac{1}{2}}) \cdot \right.\right. \\
&\left.\left. \Omega_{\bar{m}}(B^{\frac{1}{2}}(y-n\omega)) \cdot \sum_{l=-\infty}^{\infty} g(n,l) e^{2\pi il\alpha} \cdot \Omega_{m_1}(B^{\frac{1}{2}}y) dy \right| \right) \cdot (m_1+1) \le \\
&\le \sum_{l=-\infty}^{\infty} \sum_{\bar{m}=0}^{\infty} \left| I(m,\bar{m},lB^{-\frac{1}{2}}) \right| \cdot \frac{const\ (V_1)}{l^7+1} \cdot \\
&\sum_{m_1=0}^{\infty} \sum_{n=-\infty}^{\infty} e^{-\frac{\delta}{12}|n|} \left| \int_{-\infty}^{\infty} \Omega_{\bar{m}} \left(B^{\frac{1}{2}}(y-n\omega)\right) \cdot \right. \\
&\left. \cdot \Omega_{m_1}\left(B^{\frac{1}{2}}y\right) dy \right| \cdot (m_1+1) \le \\
&\le \sum_{l=-\infty}^{\infty} \sum_{\bar{m}=0}^{\infty} \left| I(m,\bar{m},lB^{-\frac{1}{2}}) \right| \cdot \frac{const\ (V_1)}{l^7+1} \cdot \left(\bar{m}^{\frac{3}{2}}+1\right) \cdot const_{10}.
\end{aligned} \tag{4.15}$$

Here the last inequality follows from (4.14). Using the result of Lemma 7 a) once more one can show that the r.h.s. of (4.15) is less than

$$const_{11} \cdot \sum_{l=-\infty}^{\infty} \left(m^2+l^4+1\right) \cdot \frac{1}{l^7+1} \le const_{12}(m^2+1).$$

The estimates of

$$\sum_{m_1=0}^{\infty}\left(\sum_{n=-\infty}^{\infty} e^{\frac{2}{3}\delta|n|}\sup_{\alpha}\left|\int_{-\infty}^{\infty}\Omega_m\left(B^{\frac{1}{2}}(y-n\omega)\right)\cdot\right.\right.$$
$$\left.\left.\cdot\frac{\partial^k}{\partial y^k}V_1^{(n)}(y+\alpha)\Omega_{m_1}\left(B^{\frac{1}{2}}y\right)dy\right|\right)\cdot(m_1^l+1)$$

for $k=1,2;\ l=1$ or $k=0,1,2;\ l>1$ can be derived in a similar way.

Proposition 4 is proven.

5. Proof of Theorem 3

Mainly we will consider the case of functions V_0, V_1 satisfying the condition (**C**). We proved in Sects. 3, 4 the existence of a unitary operator $U(p) = e^{iW}$ that $U(p)^{-1}L_{\varepsilon_0 p}U(p)\big|_{E^{(m)}_{\varepsilon_0,p}}$ is given by formulas (2.7–2.9). The columns of the matrix representation of $U(p)$ produce the new basis

$$\{e_j(p)\}_{j=-\infty}^{\infty}: e_j(p)(m,n) = e^{iW}(0,m;n-j;p+j\omega).$$

It follows from (2.3) and inductive assumptions $(II_{s,1}),\ s=1,2,3,\cdots$, that

$$|e_j(p)(m,n)|\cdot(m^2+1)e^{\frac{2}{3}\delta|n-j|} < const. \tag{5.1}$$

The last inequality, combined with the results a),b), concerning the spectrum of

$$L^{(m)}_{\varepsilon_0} = U(p)^{-1}L_{\varepsilon_0,p}U(p)\big|_{E^{(m)}_{\varepsilon_0,p}}$$

gives us the series representation (2.10–2.11) for the generalized functions of $L_{\varepsilon_0}(B)$. The trivial estimate

$$|\Omega_l(y)| < const\cdot(l+1)$$

and the formula $\frac{d}{dy}\Omega_l = \left(\frac{l}{2}\right)^{\frac{1}{2}}\Omega_{l-1} - \left(\frac{l+1}{2}\right)^{\frac{1}{2}}\Omega_{l+1}$ imply the uniform convergence of (2.11) and allow us to differentiate it twice in x and y term by term. To prove (2.12) we decompose the series (2.11) into two parts:

$$\Phi_{p,k}(x,y) = \sum_{l<\left(\frac{y}{20}\right)^2} + \sum_{l\geq\left(\frac{y}{20}\right)^2}.$$

We derive the trivial bound of the second sum

$$\left|\sum_{l\geq\left(\frac{y}{20}\right)^2}\sum_n c(l,n-k;p+k\omega)\cdot e^{2\pi i\left(\frac{p}{\omega}+n\right)x}\Omega_l\left(B^{\frac{1}{2}}(y-p-n\omega)\right)\right| \leq$$
$$\leq \frac{1}{\left(\frac{y}{20}\right)^2+1}\cdot\sum_{l,n}|c(l,n-k,p+k\omega)|\,const(l^2+1)\cdot e^{\frac{\delta}{3}|n|} \leq \frac{const(p,k)}{y^2+1}.$$

To consider the first sum, we recall that $\Omega_l(y)$ oscillates on the interval $\left[-2\sqrt{l},\ 2\sqrt{l}\right]$ and decays superexponentially off this interval. In particular

$$|\Omega_l(y)| < \exp\left(-\frac{y^2}{10}\right) \quad \text{if} \quad |y| > \sqrt{l} \cdot 10 \tag{5.2}$$

(see [22]). Thus

$$\sum_{l:|y|>20\sqrt{l}} \sum_{n} c(l, n-k, p+k\omega) \cdot e^{2\pi i\left(\frac{p}{\omega}+n\right)x} \Omega_l\left(B^{\frac{1}{2}}(y-p-n\omega)\right) =$$
$$= \sum_{l:|y|>20\sqrt{l}} \sum_{n:|n|<\frac{y}{2}} + \sum_{l:|y|>20\sqrt{l}} \sum_{n:|n|\geq\frac{y}{2}}$$

Using (5.2) in the first subsum and (2.10) in the second, we can easily show that they are exponentially small in y. This gives us (2.12). If V_0, V_1 satisfy ($\mathbf{C}^*$) we replace (5.1) by

$$|e_j(p)(m,n)| \left(m^{N+1}+1\right) e^{\frac{2}{3}\delta|n-j|} < const, \tag{5.1'}$$

where N can be taken arbitrary large if $\varepsilon_0, \varepsilon_1 \to 0$, and use similar arguments. The infinite differentiability of $\Phi^{(m)}_{p,k}$ follows from the Friedrichs theorem for strongly elliptic operators ([23]). To prove part (iii), we consider for every $k \in \mathbb{Z}^1$ and a.e. $p \in [0, \omega]$, the eigenspace $H_{m,k}(p)$ of the operator $L^{(m)}_{\varepsilon_0}(p)$, generated by the eigenfunction $\varphi_k(p)$ with the eigenvalue $\Lambda^{(m)}(p+k\omega)$. If we define $H_{m,k} = \bigoplus\int_0^\omega H_{m,k}(p)dp$, then $E^{(m)}_{\varepsilon_0} = \bigoplus\sum_{k=-\infty}^{\infty} H_{m,k}$, each $H_{m,k}$ is $L_{\varepsilon_0}(B)$ - invariant and the restriction of $L_{\varepsilon_0}(B)$ to $H_{m,k}$ is unitary equivalent to the multiplication operator on $L^2([0,\omega])$ with the multiplication function $\Lambda(\cdot + k\omega)$.

Part (ii) follows from the representation of $L_{\varepsilon_0}(B)$ as the direct integral of the difference operators, Theorem 2 and the previous considerations. Theorem 3 is proven.

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