Determinantal Random Fields

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1 Introduction

The theory of random point fields has its origins in such diverse areas of science as life tables, particle physics, population processes and communication engineering. A standard reference to the subject is the monograph by D.J. Daley and D.Vere-Jones [2].

This article is concerned with a special class of random point fields, introduced by Macchi in the mid 1970s. The model that Macchi considered describes the statistical distribution of a fermion system in thermal equilibrium. Macchi proposed to call the new class of random point processes the fermion random point processes. The characteristic property of this family of random point processes is the condition that $k-$point correlation functions have the form of determinants built from a correlation kernel. This implies that the particles obey the Pauli exclusion principle. Till mid 1990s, fermion random point processes attracted only a limited interest in mathematical and physical communities, with the exception of two important works by Spohn (1987) and Costin-Lebowitz (1995). This situation changed dramatically at the end of the last century, as the subject greatly benefited from the newly discovered connections to random matrix theory, representation theory, random growth models, combinatorics and number theory. Things are rapidly developing at the moment. Even the terminology has not yet set in stone. Many experts nowdays use the term "determinantal random point fields" instead of "fermion random point fields". We follow this trend in our article.

This article is intended as a short introduction to the subject. The next section builds a mathematical framework and gives a formal mathematical definition of the determinantal random point fields. Section 3 is devoted to the examples of determinantal random point fields from quantum mechanics, random matrix theory, random growth models, combinatorics and representation theory. In Sections 4 we discuss the ergodic properties of translation-invariant determinantal random point fields. In Sections 5 we discuss the Gibbsian property of determinantal random point fields. Central Limit Theorem type results for the counting functions and similar linear statistics is discussed in Section 6. Section 7 is devoted to some generalizations of determinantal point fields, namely immanantal and pfaffian random point fields.

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2 Mathematical Framework

We start by building a standard mathematical framework for the theory of random point processes. Let E be a one-particle space and X be a space of finite or countable configurations of particles in E. In general, E can be a separable Hausdorff space. However, for our purposes it suffices to consider $E = \mathbb{R}^d$ or $E = \mathbb{Z}^d$. We usually assume in this section that $E = \mathbb{R}^d$, with the understanding that all constructions can be easily extended to the discrete case. We assume that each configuration $\xi = (x_i),\ x_i \in E,\ i \in \mathbb{Z}^1$ (or $i \in \mathbb{Z}^1_+$ for $d > 1$), is locally finite. In other words, for every compact $K \subset E$ the number of particles in K, $\#_K(\xi) = \#(x_i \in K)$ is finite.

In order to introduce a σ-algebra of measurable subsets of X, we first define the cylinder sets. Let $B \subset E$ be a bounded Borel set and let $n \geq 0$. We call $C_n^B = \{\xi \in X : \#_B(\xi) = n\}$ a cylinder set. We define $\mathcal{B}$ as a σ-algebra generated by all cylinder sets (i.e., $\mathcal{B}$ is the minimal σ-algebra that contains all C_n^B).

Definition 1. A random point field is a triplet $(X, \mathcal{B}, \Pr)$, where $\Pr$ is a probability measure on $(X, \mathcal{B})$.

It was observed in 1960-1970s (see e.g. Lenard 1973, 1975), that in many cases the most convinient way to define a probability measure on $(X, \mathcal{B})$ is via the point correlation functions. Let $E = \mathbb{R}^d$, equipped with the underlying Lebesgue measure.

Definition 2. Locally integrable function $\rho_k : E^k \to \mathbb{R}^1_+$ is called a k-point correlation function of the random point field $(X, \mathcal{B}, \Pr)$ if, for any disjoint bounded Borel subsets $A_1, \ldots, A_m$ of E and for any $k_i \in \mathbb{Z}^1_+$, $i = 1, \ldots m$, $\sum_{i=1}^{m} k_i = k$, the following formula holds:

$$\mathbb{E} \prod_{i=1}^{m} \frac{(\#_{A_i})!}{(\#_{A_i} - k_i)!} = \int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho_k(x_1, \ldots, x_k) dx_1 \ldots dx_k, \tag{1}$$

where by $\mathbb{E}$ we denote the mathematical expectation with respect to $\Pr$. In particular, $\rho_1(x)$ is the particle density, since

$$\mathbb{E}\#_A = \int_A \rho_1(x) dx$$

for any bounded Borel $A \subset E$. In general, $\rho_k(x_1, \ldots, x_k)$ has the following probabilistic interpretation. Let $[x_1, x_i + dx_i]$, $i = 1, \ldots, k$ be infinitesimally small boxes around x_i, then $\rho_k(x_1, x_2, \ldots, x_k) dx_1 \cdot \ldots \cdot dx_k$ is the probability to find a particle in each of these boxes.

In the discrete case $E = \mathbb{Z}^d$ the construction of a random point field is very similar. The probability space X and the $\sigma-$algebra $\mathcal{B}$ are constructed essentially in the same way as before. Moreover, in the discrete case the set of the countable configurations of particles can be identified with the set of all subsets of E. Therefore $X = \{0, 1\}^E$, and $\mathcal{B}$ is generated by the events $\{C_x,\ x \in E\}$, where $C_x = \{x \in \xi\}$. The k-point correlation function $\rho(x_1, \ldots, x_k)$ is then just a probability that a configuration ξ contains the sites $x_1, \ldots, x_k$. In other words, $\rho_k(x_1, \ldots, x_k) = \Pr(\bigcap_{i=1}^k C_{x_i})$. In particular, the one-point correlation function $\rho_1(x),\ x \in \mathbb{Z}^d$, is the probability that a configuration contains the site x, i.e. $\rho_1(x) = \Pr(C_x)$.

The problem of the existence and the uniqueness of a random point field defined by its correlation functions was studied by Lenard (1973-1975). It is not surprising, that Lenard's papers revealed many parallels to the classical moment problem. In particular, the random point field is uniquely defined by its correlation functions if the distribution of random variables $\{\#_A\}$ for bounded Borel sets A is uniquely determined by its moments.

In our paper we study a special class of random point fields introduced by Macchi in [8]. To shorten the exposition, we give the definitions only in the continuous case $E = \mathbb{R}^d$. In the discrete case, the definitions are essentially the same.

Let $K : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be an integral locally trace class operator. The last condition means that for any compact $B \subset \mathbb{R}^d$ the operator $K\chi_B$ is trace class, where $\chi_B(x)$ is an indicator of B. The kernel of K is defined up to a set of measure zero in $\mathbb{R}^d \times \mathbb{R}^d$. For our purposes, it is convenient to choose it in such a way that for any bounded measurable B and any positive integer n

$$\mathrm{Tr}((\chi_B K \chi_B)) = \int_B K(x,x)dx \tag{2}$$

We refer the reader to [11], p.927 for the discussion. We are now ready to define a determinantal (fermion) random point field on $\mathbb{R}^d$.

Definition 3. A random point field on E is said to be determinantal (or fermion) if its n-point correlation functions are of the form

$$\rho_n(x_1, \ldots, x_n) = \det\big(K(x_i, x_j)\big)_{1 \le i \le n} \tag{3}$$

Remark 1. *If the kernel is Hermitian-symmetric, then the non-negativity of n-point correlation functions implies that the kernel $K(x,y)$ is non-negative definite and, therefore K must be a non-negative operator. It should be noted, however, that there exist determinantal random point fields corresponding to non-Hermitian kernels (see an example (18) in Section 3). The kernel $K(x,y)$ is usually called a correlation kernel of the determinantal random point process.*

In the Hermitian case, the necessary and sufficient conditions on the operator K to define a determinantal random point filed were established by Soshnikov ([11]; see also [8]).

Theorem 2.1 *Hermitian locally trace class operator K on $L^2(E)$ determines a determinantal random point field if and only if $0 \le K \le 1$ (in other words both K and $1 - K$ are non-negative operators). If the corresponding random point field exists, it is unique.*

The main technical part of the proof is the following proposition.

Proposition 1 *Let (X, B, P) be a determinantal random point field with the Hermitian-symmetric correlation kernel K. Let f be a non-negative continuous function with compact support. Then*

$$\mathbb{E}e^{\langle \xi, f \rangle} = \det\left(Id - (1 - e^{-f})^{1/2} K (1 - e^{-f})^{1/2}\right), \tag{4}$$

where $\langle \xi, f \rangle$ is the value of the linear statistics defined by the test function f on the configuration $\xi = (x_i)$, in other words $\langle \xi, f \rangle = \sum_i f(x_i)$.

Remark 2 . *The r.h.s. of (4) is well defined as the Fredholm determinant of a trace class operator. Letting $f = \sum_{i=1}^k s_i \chi_{I_i}$, one obtains $\mathbb{E}e^{\langle \xi, f \rangle} = \mathbb{E}\prod_{i=1}^k z_i^{\#_{I_i}}$, with $z_i = e^{s_i}$. In this case, the l.h.s. of (4) becomes the generating function of the joint distribution of the counting random variables $\#_{I_i}$, $i = 1, \ldots, k$.*

Unfortunately, there are very few known results in the non-Hermitian case. In particular, the necessary and sufficient condition on K for the existence of the determinantal random point field with the non-Hermitian correlation kernel is not known.

We finish this section with the introduction of the Janossy densities (a.k.a. density distributions, exclusion probability densities, etc) of a random point field.

The term Janossy densities in the theory of random point processes was introduced by Srinvasan in 1969, who referred to the 1950 paper by Janossy on particle showers. Let us assume that all point correlation functions exist and locally integrable, and I be a bounded Borel subset of $\mathbb{R}^d$. Intuitively, one can think of the Janossy density $\mathcal{J}_{k,I}(x_1,\dots,x_k),\quad x_1,\dots,x_k\in I$ as

$$\mathcal{J}_{k,I}(x_1,\dots,x_k)\prod_{i=1}^k dx_i = \quad \Pr\{\text{ there are exactly } k \text{ particles in } I \text{ and there is a particle in each of the } k \text{ infinitesimal boxes } (x_i, x_i+dx_i),\quad i=1,\dots,k\}. \tag{5}$$

To give a formal definition, we express point correlation functions in terms of Janossy densities and vice versa:

$$\rho_k(x_1,\dots,x_k)=\sum_{j=1}^{\infty}\frac{1}{j!}\int_{I^j}\mathcal{J}_{k+j,I}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+j})dx_{k+1}\dots dx_{k+j}, \tag{6}$$

$$\mathcal{J}_{k,I}(x_1,\dots,x_k)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\int_{I^j}\rho_{k+j}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+j})dx_{k+1}\cdots dx_{k+j} \tag{7}$$

A very useful property of the Janossy densities is that

$$\Pr\{\text{there are exactly } k \text{ particles in } I\}=\frac{1}{k!}\int_{I^k}\mathcal{J}_{k,I}(x_1,\dots,x_k)dx_1\cdots dx_k. \tag{8}$$

In the case of determinantal random point fields, Janossy densities also have a determinantal form, namely

$$\mathcal{J}_{k,I}(x_1,\dots,x_k)=\det(\mathrm{Id}-K_I)\cdot\det\big(L_I(x_i,x_j)\big)_{1\le i,j\le k}. \tag{9}$$

In the last equation, K_I is the restriction of the operator K to the $L^2(I)$. In other words, $K_I(x,y)=\chi_I(x)K(x,y)\chi_I(y)$, where χ_I is the indicator of I. The operator L_I is expressed in terms of K_I as $L_I=(\mathrm{Id}-K_I)^{-1}K_I$. For further results on the Janossy densities of determinantal random point processes we refer the reader to [13] and references therein.

3 Examples of Determinantal Random Point Fields

3.1 Fermion Gas

Let $H=-\frac{d^2}{dx^2}+V(x)$ be a Schrödinger operator with discrete spectrum on $L^2(E)$. We denote by $\{\varphi_\ell\}_{\ell=0}^{\infty}$ an orthonormal basis of the eigenfunctions, $H\varphi_\ell=\lambda_\ell\cdot\varphi_\ell,\ \lambda_0<\lambda_1\le\lambda_2\le\dots$. To define a Fermi gas, we consider the n^{th} exterior power of H, $\wedge^n(H):\wedge^n(L^2(E))\to\wedge^n(L^2(E))$, where $\wedge^n(L^2(E))$ is the space of square-integrable antisymmetric functions of n variables and $\wedge^n(H)=$

$\sum_{i=1}^{n}(-\frac{d^2}{dx_i^2}+V(x_i))$. The eigenstates of the Fermi gas are given by the normalized Slater determinants

$$\psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n!}}\sum_{\sigma\in S_n}(-1)^{\sigma}\prod_{i=1}^{n}\varphi_{k_i}(x_{\sigma(i)}) = \frac{1}{\sqrt{n!}}\det(\varphi_{k_i}(x_j))_{1\le i,j\le n}, \tag{10}$$

where $0\le k_1<k_2<\ldots<k_n$. A probability distributionof n particles in the Fermi gas is given by the squared absolute value of the eigenstate

$$p(x_1,\ldots,x_n) = |\psi(x_1,\ldots,x_n)|^2 = \frac{1}{n!}\det\big(\varphi_{k_i}(x_j)\big)_{1\le i,j\le n} \cdot \det\big(\overline{\varphi_{k_j}(x_i)}\big)_{1\le i,j\le n} = \frac{1}{n!}\det\big(K_n(x_i,x_j)\big)_{1\le i,j\le n}, \tag{11}$$

where $K_n(x,y)=\sum_{i=1}^{n}\varphi_{k_i}(x)\overline{\varphi_{k_i}(y)}$ is the kernel of the orthogonal projector onto the subspace spanned by the n eigenfunctions $\{\varphi_{k_i}\}$ of H. The $n-$dimensional probability distribution (11) defines a determinantal random point field with n particles. The k-point correlation functions are given by

$$\rho_k^{(n)}(x_1,\ldots,x_n) = \frac{n!}{(n-k)!}\int p_n(x_1,\ldots,x_n)dx_{k+1}\ldots dx_n = \det\big(K_n(x_1,x_j)\big)_{1\le i,j\le k}. \tag{12}$$

3.2 Random Matrix Models

Some of them most important ensembles of random matrices fall into the class of determinantal random point processes.

The archetypal ensemble of Hermitian random matrices is a so-called Gaussian Unitary Ensemble (GUE for short). Let us consider the space of $n\times n$ Hermitian matrices $\{A=(A_{ij})_{1\le i,j\le n},\ \mathrm{Re}(A_{ij})=\mathrm{Re}(A_{ji}),\ \mathrm{Im}(A_{ij})=-\mathrm{Im}(A_{ji})\}$. A G.U.E. random matrix is defined by its probability distribution

$$P(dA)=\mathrm{const}_n\cdot\exp(-\mathrm{Tr}A^2)dA, \tag{13}$$

where dA is a Lebesgue measure, i.e., $dA=\prod_{i<j}d\mathrm{Re}(A_{ij})d\mathrm{Im}(A_{ij})\prod_{k=1}^{n}dA_{kk}$. The eigenvalues of a random Hermitian matrix are real random variables, whose joint probability distribution is a determinantal random point process of n particles on the real line. The correlation kernel has the Christoffel-Darboux form built from the Hermite polynomials.

The G.U.E. ensemble of random matrices is invariant under the unitary transformation $A\to UAU^{-1}, U\in U(n)$. An important generalization of (13) that preserves the unitary invariance is

$$P(dA)=\mathrm{const}_n\exp(-\mathrm{Tr}V(A))dA \tag{14}$$

where, for example, $V(x)$ is a polynomial of even degree with a positive leading coefficients. The correlation functions of the eigenvalues in (14) are again determinantal, and the Hermite polynomials in the correlation kernel have to be replaced by the orthonormal polynomials with respect to the weight $\exp(-V(x))$. For the details, we refer the reader to the monographs by Mehta (2004) and Deift(2000).

There are many other ensembles of random matrices for which the joint distribution of the eigenvalues has determinantal point correlation functions: classical compact groups with respect to the Haar measure, complex non-Hermitian Gaussian random matrices, positive Hermitian random matrices of the Wishart type, chains of correlated Hermitian matrices. We refer the reader to [11] for more information.

3.3 Discrete Translation-Invariant Determinantal Random Point Fields

Let $g : \mathbb{T}^d \to [0,1]$ be a Lebesgue-measurable function on the $d-$dimensional torus $\mathbb{T}^d$. Assume that $0 \le g \le 1$. A configuration ξ in $\mathbb{Z}^d$ can be though of as a $0-1$ function on $\mathbb{Z}^d$, i.e. $\xi(x) = 1$ if $x \in \xi$ and $\xi(x) = 0$ otherwise. We define a $\mathbb{Z}^d-$ invariant probability measure Pr on the Borel sets of $X = \{0,1\}^{\mathbb{Z}^d}$in such a way that

$$\rho_k(x_1, \dots, x_k) = \Pr\left(\xi(x_1) = 1, \dots, \xi(x_k) = 1\right) := \det\left(\hat{g}(x_i - x_j)\right)_{1\le i,j\le k}, \tag{15}$$

for $x_1, \dots, x_k \in \mathbb{Z}^d$. In the above formula, $\{g(n)\}$ are the Fourier coeficients of g, i.e. $g(x) = \sum_n \hat{g}(n) e^{in\cdot x}$. It is clear from Definition 3, that (15) defines a determinantal random point field on $\mathbb{Z}^d$ with the translation-invariant kernel $K(x,y) = \hat{g}(x-y)$. Below we discuss several examples that fall into this category. For the further discussion we refer the reader to [7] and [11].

a) In the trivial case when g is identically a constant $p \in [0,1]$, we obtain the i.i.d. Bernoulli(p) probability measure.

b) The edges of the uniform spanning tree in $\mathbb{Z}^2$ parallel to the horizontal axis can be viewed as the determinantal random point field in Z^2 with $g(x,y) = \frac{\sin^2 \pi x}{\sin^2 \pi x + \sin^2 \pi y}$. Similarly, the edges of the uniform spanning forest in Z^d parallel to the x_1 axis correspond to the function $g(x_1, \dots, x_d) = \frac{\sin^2 \pi x_1}{\sum_{i=1}^d \sin^2 \pi x_i}$ (the uniform spanning forest on $\mathbb{Z}^d$ is a tree only for $d \le 4$). The result is due to Burton and Pemantle (1993).

c) Let $d = 1$ and γ be a parameter between 0 and 1. Consider $g(x) = \frac{(1-\gamma)^2}{|e^{2\pi i x} - \gamma|^2}$. The corresponding probability measure is a renewal process and $K(n) = \hat{g}(n) = \frac{1-\gamma}{1+\gamma}\gamma^{|n|}$ (see [11]).

d) The process with $g(x) = \chi_I(x)$, where I is an arbitrary arc of a unit circle, appeared in the work of Borodin, Olshanski and Okounkov (2000). The corresponding correlation kernel is known as the discrete sine kernel. The determinantal random point process on $\mathbb{Z}^1$ with the discrete sine kernel describes the typical form of large Young diagrams "in the bulk" (see the next subsection).

e) The discrete sine correlation kernel with $g = \chi_{[0,1/2]}$ appeared in the zig-zag process (Johansson, 2002) derived from the uniform domino tilings in the plane corresponds to $g = \chi_{[0,1/2]}$.

3.4 Determinantal Measures on Partitions

By a partition of $n = 1, 2, \dots$ we understand a collection of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $\lambda_1 + \dots + \lambda_m = n$ and $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_m$. We shall use a notation Par(n) for the set of all partitions of n.

The Plancherel measure M_n on the set Par(n) is defined as

$$M_n(\lambda) = \frac{(\dim \lambda)^2}{n!}, \tag{16}$$

where $\dim\lambda$ is the dimension of the corresponding irreducible representation of the symmetric group S_n. Let $\mathrm{Par} = \bigsqcup_{n=0}^{\infty} \mathrm{Par}(n)$. Consider a probability measure M^θ on Par

$$\begin{aligned} &M^\theta(\lambda) = e^{-\theta}\frac{\theta^n}{n!}M_n(\lambda), \\ &\text{where } \lambda \in \mathrm{Par}(n), n = 0, 1, 2, \dots,\ \ 0 \le \theta < \infty. \end{aligned} \tag{17}$$

M^θ is called the poissonization of the measures M_n. The analysis of the asymptotic properties of M_n and M^θ has been important in connection to the famous Ulam problem and related questions in representation theory.

It was shown by Borodin, Okounkov and Olshanski (2000), and, independently, Johansson (2001) that M^θ is a determinantal random point field. The corresponding correlation kernel K (in so called the modified Frobenius coordinates) is a so-called discrete Bessel kernel on $\mathbb{Z}^1$,

$$K(x,y) = \begin{cases} \sqrt{\theta}\frac{J_{|x|-\frac{1}{2}}(2\sqrt{\theta})J_{|y|+\frac{1}{2}}(2\sqrt{\theta}) - J_{|x|+\frac{1}{2}}(2\sqrt{\theta})J_{|y|-\frac{1}{2}}(2\sqrt{\theta})}{|x|-|y|}, & \text{if } xy > 0, \\ \sqrt{\theta}\frac{J_{|x|-\frac{1}{2}}(2\sqrt{\theta})J_{|y|-\frac{1}{2}}(2\sqrt{\theta}) - J_{|x|+\frac{1}{2}}(2\sqrt{\theta})J_{|y|+\frac{1}{2}}(2\sqrt{\theta})}{x-y}, & \text{if } xy < 0, \end{cases} \tag{18}$$

where $J_x(\cdot)$ is the Bessel function of order x. One can observe that the kernel $K(x,y)$ is not Hermitian, but the restriction of this kernel to the positive and negative semi-axis is Hermitian.

M^θ is a special case of an infinite parameter family of probability measures on Par, called the Schur measures, and defined as

$$M(\lambda) = \frac{1}{Z}s_\lambda(x)s_\lambda(y), \tag{19}$$

where s_λ are the Schur functions, $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are parameters such that

$$Z = \sum_{\lambda\in\mathrm{Par}} s_\lambda(x)s_\lambda(y) = \prod_{i,j}(1 - x_i y_j)^{-1} \tag{20}$$

is finite and $\{x_i\}_{i=1}^\infty = \overline{\{y_i\}_{i=1}^\infty}$. It was shown by Okounkov (2000), that the Schur measure belong to the class of the determinantal random point fields.

3.5 Non-Intersecting Paths of a Markov Process

Let $p_{t,s}(x,y)$ be the transition probability of a Markov process $\xi(t)$ on $\mathbb{R}$ with continuous trajectories and let $(\xi_1(t), \xi_2(t), \dots, \xi_n(t))$ be n independent copies of the process. A classical result of Karlin and McGregor (1959) states that if n particles start at the positions $x_1^{(0)} < x_2^{(0)} < \dots < x_n^{(0)}$, then the probability density of their joint distribution at time $t_1 > 0$, given that their paths have not intersected for all $0 \le t \le t_1$, is equal to

$$\pi_{t_1}(x_1^{(1)}, \dots, x_n^{(1)}) = \frac{1}{Z}\det(p_{0,t_1}(x_i^{(0)}, x_j^{(1)}))_{i,j=1}^n$$

provided the process $(\xi_1(t), \xi_2(t), \dots, \xi_n(t))$ in $\mathbb{R}^n$ has a strong Markovian property.

Let $0 < t_1 < t_2 < \dots < t_{M+1}$. The conditional probability density that the particles are in the positions $x_1^{(1)} < x_2^{(1)} < \dots < x_n^{(1)}$ at time t_1, at the positions $x_1^{(2)} < x_2^{(2)} < \dots < x_n^{(2)}$ at time t_2,..., at the positions $x_1^{(M)} < x_2^{(M)} < \ldots < x_n^{(M)}$ at time t_M, given that at time t_{M+1} they are at the positions $x_1^{(M+1)} < x_2^{(M+1)} < \ldots < x_n^{(M+1)}$ and their paths have not intersected, is then equal to

$$\pi_{t_1,t_2,\ldots,t_M}(x_1^{(1)},\ldots,x_n^{(M)}) = \frac{1}{Z_n, M}\prod_{l=0}^{M}\det(p_{t_l,t_{l+1}}(x_i^{(l)},x_j^{(l+1)}))_{i,j=1}^{n}, \tag{21}$$

where $t_0 = 0$.

It is not difficult to show that (21) can be viewed as a determinantal random point process (see e.g. [5]).

The formulas of a similar type also appeared in the papers by Johansson, Prähofer, Spohn, Ferrari, Forrester, Nagao, Katori and Tanemura in the analysis of polynuclear growth models, random walks on a discrete circle and related problems.

4 Ergodic Properties

As before let $(X, \mathcal{B}, \Pr)$ be a random point field with a one-particle space E. Hence X is a space of the locally finite configurations of particles in E, $\mathcal{B}$ is a Borel σ-algebra of measurarble subsets of X, and Pr is a probability measure on (X, B). Throughout this section we always assume $E = \mathbb{R}^d$ or $\mathbb{Z}^d$. We define an action $\{T^t\}_{t\in E}$ of the additive group E on X in the following natural way:

$$T^t : X \to X,\ (T^t\xi)_i = (\xi)_i + t. \tag{22}$$

We recall that a random point field $(X, \mathcal{B}, P)$ is called translation invariant if for any $A \in \mathcal{B}$, any $t \in E$, $\Pr(T^{-t}A) = \Pr(A)$. The translation invariance of the correlation kernel $K(x,y) = K(x-y,0) =: K(x-y)$ implies the translation invariance of k-point correlation functions

$$\rho_k(x_1+t,\ldots,x_k+t) = \rho_k(x_1,\ldots,x_k),\ \text{a.e. } k = 1,2,\ldots,t \in E, \tag{23}$$

which, in turn, implies the translation-invariance of the random point field. The ergodic properties of such point fields were studied by several mathematicians (Soshnikov, 2000; Shirai and Takahashi, 2003; Lyons and Steif, 2003). The first general result in this direction was obtained in [11].

Theorem 4.1 *Let (X, B, P) be a determinantal random point field with a translation-invariant correlation kernel. Then the dynamical system $(X, B, P, \{T^t\})$ is ergodic, has the mixing property of any multiplicity and its spectra is absolutely continuous.*

We refer the reader to the article on Ergodic Theory for the definitions of ergodicity, mixing property, asbolute continuous spectrum of the dynamical system, etc.

In the discrete case (15), $E = \mathbb{Z}^d$, more is known. Lyons and Steif (2003) proved that the shift dynamical system is Bernoulli, i.e. it is isomorphic (in the ergodic theory sense) to an i.i.d. process. Under the additional conditions $Spec(K) \subset (0,1)$ and $\sum_n |n||K(n)|^2 < \infty$, Shirai and Takahashi [9] proved the uniform mixing property.

5 Gibbsian Properties

The first who asked the question of the Gibssianan nature of the determinantal random point fields were Costin and Lebowitz (1995), who studied the continuous determinantal random point process on $\mathbb{R}^1$ with a so-called sine correlation kernel $K(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$. The first rigorous result (in the discrete case) was established by Shirai and Takahashi ([10]).

Theorem 5.1 *Let E be a countable discrete space and K be a symmetric bounded operator on $l^2(E)$. Assume that $Spec(K) \subset (0,1)$. Then (X, B, P) is a Gibbs measure with the potential U given by $U(x|\xi) = -\log\left(J(x,x) - \langle J_\xi^{-1} j_\xi^x, j_\xi^x \rangle\right)$, where $x \in E$, $\xi \in X$, $\{x\} \cap \xi = \emptyset$. Here $J(x,y)$ stands for the kernel of the operator $J = (Id - K)^{-1}K$, and we set $J_\xi = (J(y,z))_{y,z\in\xi}$ and $j_\xi^x = (J(x,y))_{y\in\xi}$.*

We recall that the Gibbsian property of the probability measure P on $(X, \mathcal{B})$ means that

$$E\left[F|\mathcal{B}_{\Lambda^c}\right](\xi) = \frac{1}{Z_{\Lambda,\xi}} \sum_{\eta\subset\Lambda} e^{-U(\eta|\xi_{\Lambda^c})} F(\eta \cup \xi_{\Lambda^c}),$$

where Λ is a finite subset of E, $\mathcal{B}_{\Lambda^c}$ is the $\sigma-$ algebra generated by the $\mathcal{B}-$ measurable functions with the support outside of Λ, $E\left[F|\mathcal{B}_{\Lambda^c}\right]$ is the conditional mathematical expectation of the integrable function F on $(X, \mathcal{B}, P)$ with respect to the $\sigma-$ algebra $\mathcal{B}_{\Lambda^c}$. The potential U is uniquely defined by the values of $U(x,\xi)$, as follows from the following recursive relation

$$U(\{x_1,\ldots,x_n\}|\xi) = U(x_n|\{x_1,\ldots,x_{n-1}\}\cup\xi) + U(x_{n-1}|\{x_1,\ldots,x_{n-2}\}\cup\xi) + \ldots + U(x_1|\xi).$$

We refer the reader to the article on Gibbs states for additional information about the Gibbsian property. In the continuous case much less is known. Some generalized form of Gibssianness, under quite restrictive conditions, was recently established by Georgii and Yoo (2004).

6 Central Limit Theorem for Counting Function

In this section we discuss Central Limit Theorem type results for the linear statistics. The first important result in this direction was established by Costin and Lebowitz in 1995, who proved the Central Limit Theorem for the number of particles in the growing box, $\#_{[-L,L]}$, $L \to \infty$, in the case of the determinantal random point process on $\mathbb{R}^1$ with the sine correlation kernel $K(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$. Below we formulate the Costin-Lebowitz theorem in its general form due to Soshnikov (1999, 2000).

Theorem 6.1 *Let E be as in (1.1), $\{0 \le K_t \le 1\}$ a family of locally trace class operators in $L^2(E)$, $\{(X, \mathcal{B}, P_t)\}$ a family of the corresponding determinantal random point fields in E, and $\{I_t\}$ a family of measurable subsets in E such that*

$$\mathrm{Var}\#_{I_t} = \mathrm{Tr}(K_t \cdot \chi_{I_t} - (K_t \cdot \chi_{I_t})^2) \to \infty \text{ as } t \to \infty. \tag{24}$$

Then the distribution of the normalized number of particles in I_t (with respect to P_t) converges to the normal law, i.e.,

$$\frac{\#_{I_t} - \mathbb{E}\#_{I_t}}{\sqrt{\mathrm{Var}\#_{I_t}}} \xrightarrow{w} N(0,1)$$

An analogous result holds for the joint distribution of the counting functions $\{\#_{I_{t^1}}, \ldots, \#_{I_{t^k}}\}$, where $I_t^1, \ldots I_t^k$ are disjoint measurable subsets in E.

The proof of the Costin-Lebowitz theorem uses the $k-$point cluster functions. In the determinantal case, the cluster function have a simple form

$$r_k(x_1, \ldots, x_k) = (-1)^l \frac{1}{l} \sum_{\sigma \in S_k} K(x_{\sigma(1)}, x_{\sigma(2)}) K(x_{\sigma(2)}, x_{\sigma(3)}) \ldots K(x_{\sigma(k)}, x_{\sigma(1)}). \tag{25}$$

The importance of the cluster function stems from the fact that the integrals of the $k-$ point cluster function over the $k-$cube with a side I can be expressed as a linear combination of the first k cumulants of the counting random variable $\#_I$. In other words,

$$\int_{I \times \ldots \times I} r_k(x_1, \ldots, x_k) dx_1 \cdots dx_k = \sum_{l=1}^{k} \beta_{kl} C_l(\#_I). \tag{26}$$

It follows from (25), that the integral at the l.h.s. of (26) equals, up to a factor $(-1)^l(l-1)!$, to the trace of the $k-$th power of the restriction of K to I. This allows one to estimate the cumulants of the counting random variable $\#_I$. For the details, we refer the reader to [11]. The Central Limit Theorem for a general class of linear statistics, under some technical assumptions on the correlation kernel was proven in [12]. Finally we refer the reader to [11] for the Functional Central Limit Theorem for the empirical distribution function of the nearest spacings.

7 Generalizations: Immanantal and Pfaffian Point Processes

In this section we discuss two important generalizations of the determinantal point processes.

7.1 Immanantal Processes

Immanantal random point processes were introduced by P.Diaconis and S.N.Evans in 2000. Let λ be a partition of n. Denote by χ^λ the character of the corresponding irreducible representation of the symmetric group S_n. Let $K(x, y)$, be a non-negative definite, Hermitian kernel. An immanatal random point process is defined through the correlation functions

$$\rho_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \prod_{i=1}^{n} K(x_i, x_{\sigma(i)}). \tag{27}$$

In other words, the correlation functions are given by the immanants of the matrix with the entries $K(x_i, x_j)$. We will denote the r.h.s. of (27) by $K^\lambda[x_1, \ldots, x_n]$.

In the special case $\lambda = (1^n)$ (i.e. λ consists of n parts, all of which equal to 1), one obtains that $\chi^\lambda(\sigma) = (-1)^\sigma$, and $K^\lambda[x_1, \ldots, x_n] = \det(K(x_i, x_j))$. Therefore, in the case $\lambda = (1^n)$ the random point process with the correlation functions (27) is a determinantal random point process. When $\lambda = (n)$ (that is the permitation has only one part, namely n) we have $\chi^\lambda = 1$ identically, and $K^\lambda[x_1, \ldots, x_n] = per(K(x_i, x_j))$, the permanent of the matrix $K(x_i, x_j)$. The corresponding random point process is known as the boson random point process.

7.2 Pfaffian Processes

Let $K(x,y) = \begin{pmatrix} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{pmatrix}$ be an antisymmetric 2×2 matrix valued kernel, i.e. $K_{ij}(x,y) = -K_{ji}(y,x)$, $i,j = 1,2$. The kernel defines an integral operator acting on $L^2(E) \bigoplus L^2(E)$, which we assume to be locally trace class. A random point process on E is called pfaffian if its point correlation functions have a pfaffian form

$$\rho_k(x_1, \ldots, x_k) = pf(K(x_i, x_j))_{i,j=1,\ldots,k}, \quad k \geq 1. \tag{28}$$

The r.h.s. of (28) is the pfaffian of the $2k \times 2k$ antisymmetric matrix (since each entry $K(x_i, x_j)$ is a 2×2 block). Determinantal random point processes is a special case of the pfaffian processes, corresponding to the matrix kernel of the form $K(x,y) = \begin{pmatrix} 0 & \tilde{K}(x,y) \\ -\tilde{K}(y,x) & 0 \end{pmatrix}$, where $\tilde{K}$ is a scalar kernel. The most well known examples of the pfaffian random point processes, that can not be reduced to determinantal form are $\beta = 1$ and $\beta = 4$ polynomial ensembles of random matrices and their limits (in the bulk and at the edge of the spectrum), as the size of a matrix goes to infinity.

8 See Also

Integrable systems in random matrix theory. Growth processes in random matrix theory. Random matrix theory in physics. Symmetry classes in random matrix theory. Young diagrams and stochastic methods. Random partitions. Quantum Chaos. Ergodic Theory. Gibbs states. Dimer problems. Statistical mechanics and combinatorial problems. Toeplitz determinants and statistical mechanics.

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